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SUBLAMINATE ANALYSIS OF INTERLAMINAR FRACTURE IN COMPOSITES \*

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Abstract

A simple analysis method based upon a transverse shear deformation theory and a sublamine approach is utilized to analyze a Mixed-Mode edge delamination specimen. The analysis provides closed form expressions for the interlaminar shear stresses ahead of the crack, the total energy release rate, and the energy release rate components. The parameters controlling the behavior are identified. The effect of specimen stacking sequence and delamination interface on the strain energy release rate components is investigated. Results are compared with a finite element simulation for reference. The simple nature of the method makes it suitable for preliminary design analyses which require a large number of configurations to be evaluated quickly and economically.

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## Introduction

Delaminations along the free edges of laminates subjected to tensile loading have been observed during testing and service. The presence of delamination, initiated by interlaminar stresses, causes redistribution of the stresses among plies in a laminate and, therefore, usually results in a reduction of stiffness and strength. The edge delamination (ED) test has been proposed by Pagano and Pipes<sup>1</sup> to characterize the interlaminar peel strength of laminated composite materials. O'Brien<sup>2</sup> extended the scope of the test to investigate delamination onset and growth in graphite/epoxy laminates under uniform extension. A simple expression was also developed for the total energy release rate. The energy release rate components  $G_I$ ,  $G_{II}$  and  $G_{III}$  associated with the opening, shearing and tearing modes, respectively, were estimated based on a finite element simulation and the crack-closure method<sup>2,3</sup>.

A similar approach was used to study delaminations around an open hole in composite laminates<sup>4</sup>. Discrete locations around the hole boundary were modeled as straight edges, with the ply orientations rotated by an appropriate angle. Delamination was found to be governed by the percentage of Mode I for a given geometry under static loading for the graphite/epoxy material systems under consideration.

Whitney and Knight<sup>5</sup> developed an ED specimen which produces Mode I behavior. The analysis was based on classical laminated plate theory and continuity of displacements, force resultants, and moment resultants between the cracked and uncracked regions of the plate were not satisfied. In addition, such an approach precludes any reasonable determination of the effect of specimen geometry on  $G_I$ .

Recently<sup>6</sup> Whitney developed a higher order laminated plate theory which includes transverse shear deformation and a thickness-stretch mode to analyze a Mode I ED specimen. The effect of specimen geometry on strain energy release rate was also investigated for 12 ply laminates of the class  $[\theta/-\theta_2/\theta/90_2]_s$  and for the class  $[0_3/90_3]_s$ .

It is the purpose of the present work to develop a simple model for the analysis of Mixed-Mode ED specimens. Such a model provides closed-form estimates of  $G_I$ ,  $G_{II}$  and  $G_{III}$  and hence allows one to establish appropriate failure criteria<sup>7,8</sup> for delamination.

#### Preliminary Remarks

Consider the ED specimen shown in Figure 1 subjected to a uniform strain  $\epsilon_x = \epsilon$ . Due to symmetry one quarter of the laminate is analyzed as shown in Figure 2. The response is only a function of Y and Z. The laminate is divided into sublaminates with thicknesses  $h_1$  and  $h_0$  and local coordinate systems  $z_1, y_1$  and  $z_0, y_0$ . The crack length is denoted by  $a$ . Sublaminates 1 and 2, and 0 and 3 represent the groups of plies above and below the interface along which delamination occurs, respectively.

In order to provide an accurate estimate of interlaminar stresses, a higher-order theory should be considered since classical laminated plate theory predicts zero interlaminar stresses. A shear deformation theory can be used for this purpose. This theory provides a good estimate for interlaminar shear stresses  $\tau_{xz}$  and  $\tau_{yz}$ . However, the interlaminar peel stress  $\sigma_{zz}$  is not accurate. The reason for that is the absence of thickness strain.

From symmetry, the transverse displacement  $w$  is zero at  $Z=0$ , hence the prescription of  $w$  at the middle plane  $Z=0$  fixes  $w$  everywhere. In this case, the vertical shearing force resultant at both ends cannot be prescribed and the distribution of the peel stress will not be correct. In spite of this simplification, reliable energy release rate components can be estimated based on interlaminar shear stresses.  $G_I$  is evaluated as  $G_T - (G_{II} + G_{III})$ , where the total energy release rate is denoted by  $G_T$ .

In the present formulation, thickness strain is neglected and consequently considerable simplification in the analysis is achieved.

Another source of simplification in the present approach is due to the modeling of the structure as sublaminae---group of plies that are conveniently treated as laminated units. This approach can be applied with confidence if the characteristic length of the response is large compared to the individual sublaminate thickness.<sup>9</sup>

### Overview of the Analytical Solution

In the following sections a step by step procedure is provided for the solution of the ED specimen. Intermediate results are also provided. The governing equations are derived in Appendix I. Expression for the interlaminar stresses, total energy release rate and energy release rate components are given in Equations (34), (40) and (42)-(43), respectively. The parameters associated with these equations are provided explicitly in terms of the stiffness coefficients in Appendix II.

A solution based on Classical Lamination Theory is given in Appendix III. This solution represents the behavior in the interior of the laminate. Application of the present analysis and comparison with a finite

element simulation for 63 test cases is presented under the section entitled Results and Discussion.

The reader interested in results and comparison can refer directly to the "Results and Discussion" section on page 28.

### Analysis

Assume the following displacement field within each sublaminar:

$$\begin{aligned} u &= x\epsilon + U(y) + z\beta_x(y) \\ v &= V(y) + z\beta_y(y) \\ w &= W(y) \end{aligned} \tag{1}$$

where  $u, v$ , and  $w$  denote displacements relative to the  $x, y$ , and  $z$  axes, respectively, and  $\epsilon$  is a uniform axial strain. Coordinates  $y$  and  $z$  are local coordinates as shown in Figure 2. The present formulation recognizes shear deformation through the rotations  $\beta_x$  and  $\beta_y$ . The corresponding strains are

$$\begin{aligned} \epsilon_{xx} &= \epsilon \\ \epsilon_{yy} &= V_{,y} + z\beta_{y,y} \\ \epsilon_{zz} &= 0 \\ \gamma_{xy} &= U_{,y} + z\beta_{x,y} \\ \gamma_{xz} &= \beta_x \\ \gamma_{yz} &= \beta_y + W_{,y} \end{aligned} \tag{2}$$

The variables associated with sublaminae 0 through 3 will be written with subscripts 0 through 3, respectively.

From symmetry

$$\begin{aligned} w_o(y, -h_o/2) &= 0 \\ \text{or} \quad w_o &= 0 \end{aligned} \quad (3)$$

From continuity of displacements at the interface between sublamine (1) and (0)

$$\begin{aligned} u_o(y, h_o/2) &= u_1(y, -h_1/2) \\ v_o(y, h_o/2) &= u_1(y, -h_1/2) \\ w_o(y, h_o/2) &= w_1(y, -h_1/2) \end{aligned} \quad (4)$$

Substitute from Equation (1) into (4) to get

$$U_o = U_1 - \frac{h_1}{2} \beta_{1x} - \frac{h_o}{2} \beta_{ox}$$

$$V_o = V_1 - \frac{h_1}{2} \beta_{1x} - \frac{h_o}{2} \beta_{ox}$$

$$W_1 = W_o = 0 \quad (5)$$

### Governing Equations

The governing equations for each sublamine are derived in Appendix I using a virtual work approach. These equations are written below for convenience.

### Constitutive Relationship

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \epsilon \\ V_{,y} \\ U_{,y} \\ \beta_{y,y} \\ \beta_{x,y} \end{bmatrix}$$

$$\begin{bmatrix} Q_y \\ Q_x \end{bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{bmatrix} \beta_y + W_{,y} \\ \beta_x \end{bmatrix}$$

(6)

### Equilibrium Equations

$$\begin{aligned}
 N_{xy,y} + n_x &= 0 \\
 N_{y,y} + n_y &= 0 \\
 Q_{y,y} + q &= 0 \\
 M_{xy,y} - Q_x + m_x &= 0 \\
 M_{y,y} - Q_y + m_y &= 0
 \end{aligned}$$

(7)

where  $n_x$ ,  $n_y$ ,  $q$ ,  $m_x$  and  $m_y$  are defined in Equation (I-9) of Appendix I.

The equilibrium equations can be written in terms of kinematic variables by substitution from Equation (6) into Equation (7). The result is

$$\begin{bmatrix}
 A_{66}L_{yy} & & & & \\
 A_{26}L_{yy} & A_{22}L_{yy} & & & \\
 0 & 0 & -A_{44}L_{yy} & & \\
 B_{66}L_{yy} & B_{26}L_{yy} & -A_{45}L_y & (D_{66}L_{yy} - A_{55}) & \\
 B_{26}L_{yy} & B_{22}L_{yy} & -A_{44}L_y & (D_{26}L_{yy} - A_{45}) & (D_{22}L_{yy} - A_{44})
 \end{bmatrix}
 \begin{bmatrix}
 U \\
 V \\
 W \\
 \beta_x \\
 \beta_y
 \end{bmatrix}
 = -
 \begin{bmatrix}
 n_x \\
 n_y \\
 -q \\
 m_x \\
 m_y
 \end{bmatrix}$$

$$L_{yy} = d^2/dy^2$$

$$L_y = d/dy$$

(8)

### Solution Methodology

The above collection of equations are to be applied to individual plies of a laminate or to groups of plies... sublaminate. The solution steps are summarized in the following:

1. Divide the laminate into sublaminate according to geometry and loading condition. The sublaminate length is selected such that within the sublaminate the geometry and loading are continuous as is commonly done in engineering analysis of simple structures.
2. The displacements, resultant forces and moments, and interlaminar stresses in each sublaminate are governed by the equilibrium Equation(7), the constitutive relations (6) and the displacement distributions (1). Write these equations for each sublaminate in the analysis model.
3. Apply interlaminar continuity conditions and enforce traction or displacement conditions at the extreme upper and lower surfaces of the laminate.



4. Solve the system of coupled ordinary differential equations for the element variables.
5. Enforce the boundary conditions at constant values of  $y$ , the laminate sections, as well as continuity requirements between sublaminates ends in order to find the values of the arbitrary constants resulting from the solution in step 4.
6. Determine interlaminar stresses, resultant forces and moments displacement distribution and energy release rate.

#### Application to the ED Specimen

The ED configuration is divided into four sublaminae as shown in Figure 2. The response associated with sublaminae 1 and 0 is coupled through the continuity conditions at their common interface. Hence, the variables associated with both sublaminae are to be solved simultaneously. The situation is different with sublaminae 2 and 3 where the continuity conditions are relaxed due to the presence of the crack. Therefore, the variables associated with these sublaminae are not coupled.

The solution procedure for sublaminae 1 and 0--the uncracked portion of the laminate---is presented first, followed by the cracked portion represented by sublaminae 2 and 3.

Uncracked Region of the Laminate:

##### (i) Sublaminar 1

The upper surface of this sublaminar is stress free. Denote the shear and peel stress at the bottom surface by  $t_x$ ,  $t_y$  and  $p$ , respectively. The transverse displacement  $w_1$ , is zero from Equation (5). Hence, the equilibrium equation in terms of the displacement variable takes the form

$$\begin{bmatrix}
A_{66}^1 L_{yy} & A_{26}^1 L_{yy} & B_{66}^1 L_{yy} & B_{26}^1 L_{yy} \\
A_{26}^1 L_{yy} & A_{22}^1 L_{yy} & B_{26}^1 L_{yy} & B_{22}^1 L_{yy} \\
0 & 0 & -A_{45}^1 L_y & -A_{44}^1 L_y \\
B_{66}^1 L_{yy} & B_{26}^1 L_{yy} & (D_{66}^1 L_{yy} - A_{55}^1)(D_{26}^1 L_{yy} - A_{45}^1) \\
B_{26}^1 L_{yy} & B_{22}^1 L_{yy} & (D_{26}^1 L_{yy} - A_{45}^1)(D_{22}^1 L_{yy} - A_{44}^1)
\end{bmatrix}
\begin{bmatrix}
U_1 \\
V_1 \\
\beta_{1x} \\
\beta_{1y}
\end{bmatrix}
= -
\begin{bmatrix}
-t_x \\
-t_y \\
p \\
\frac{h_1}{2} t_x \\
\frac{h_1}{2} t_y
\end{bmatrix}
\quad (9)$$

(ii) Sublamine 0

From symmetry condition at the sublamine bottom surface the shear stresses  $t_{1x}$  and  $t_{1y}$  are zero. From reciprocity of stresses at the interface between sublaminates 0 and 1, the interlaminar stresses at the upper surface of sublamine 0 are  $t_x$ ,  $t_y$  and  $p$ . Denote the peel stress at the sublamine bottom surface by  $p_1$ . Hence the equilibrium equation takes the form

$$\begin{bmatrix}
A_{66}^0 L_{yy} & A_{26}^0 L_{yy} & B_{66}^0 L_{yy} & B_{26}^0 L_{yy} \\
A_{26}^0 L_{yy} & A_{22}^0 L_{yy} & B_{26}^0 L_{yy} & B_{22}^0 L_{yy} \\
0 & 0 & -A_{45}^0 L_y & -A_{44}^0 L_y \\
B_{66}^0 L_{yy} & B_{26}^0 L_{yy} & (D_{66}^0 L_{yy} - A_{55}^0)(D_{26}^0 L_{yy} - A_{45}^0) \\
B_{26}^0 L_{yy} & B_{22}^0 L_{yy} & (D_{26}^0 L_{yy} - A_{45}^0)(D_{22}^0 L_{yy} - A_{44}^0)
\end{bmatrix}
\begin{bmatrix}
U_0 \\
V_0 \\
\beta_{0x} \\
\beta_{0y}
\end{bmatrix}
= -
\begin{bmatrix}
t_x \\
t_y \\
p_1 - p \\
\frac{h_0}{2} t_x \\
\frac{h_0}{2} t_y
\end{bmatrix}$$

(10)

Equations (9), (10) and (5) can be combined to yield the following form:

$$\begin{bmatrix} (F_{11}L_{yy}-A_{44}^1) & & & \\ F_{21}L_{yy} & (F_{22}L_{yy}-A_{44}^0) & & \\ (F_{31}L_{yy}-A_{45}^1) & F_{32}L_{yy} & (F_{33}L_{yy}-A_{55}^1) & \\ F_{41}L_{yy} & (F_{42}L_{yy}-A_{45}^0) & F_{43}L_{yy} & (F_{44}L_{yy}-A_{55}^0) \end{bmatrix} \begin{bmatrix} \beta_{1y} \\ \beta_{oy} \\ \beta_{1x} \\ \beta_{ox} \end{bmatrix} = 0 \quad \text{SYM} \quad (11)$$

The parameters in Equation (11) are defined in Equation (II-1) of Appendix II.

Assume an exponential solution of the form

$$(\beta_{1y}, \beta_{oy}, \beta_{1x}, \beta_{ox}) = (\beta_{1y}^*, \beta_{oy}^*, \beta_{1x}^*, \beta_{ox}^*)e^{sy} \quad (12)$$

Substitute from Equation (12) into Equation (11) to get the following characteristic equation

$$E_8s^8 + E_6s^6 + E_4s^4 + E_2s^2 + E_0 = 0 \quad (13)$$

Coefficients  $E_8$  through  $E_0$  are defined in Equation (II-5) of Appendix II. Parameter  $E_8$  represents the determinant of the coefficients of  $L_{yy}$  in matrix equation (11) while  $E_0$  is the determinant when  $L_{yy}$  is set to zero. Parameter  $E_0$  depends solely on the stiffness coefficients  $A_{44}$ ,  $A_{55}$  and  $A_{45}$  for both sublaminae while  $E_8$  is predominantly influenced by the bending and coupling coefficients  $D_{ij}$  and  $B_{ij}$ . Hence its numerical value can be orders of magnitude smaller than the remaining coefficients. This fact results in the presence of a boundary zone in the response.

Typical values of a nondimensional form of the coefficients  $E_8$  through  $E_0$  is given in Table II for three laminates made of T300/5208 graphite/epoxy material. The nondimensional form of the coefficients is obtained by making the following substitution

$$\bar{s} = sb \quad (14)$$

Where  $b$  is the laminate semi-width. The arrows in the layups of Table II indicate the interfaces containing delaminations. The material properties and geometry appear in Table I.

The characteristic roots controlling the behavior are determined from Equation (13), which has a closed-form solution. However when  $A_{j6}$ ,  $B_{j6}$ ,  $D_{j6}$  ( $j = 1, 2$ ) and  $A_{45}$  are neglected, Equation (11) takes an uncoupled form and consequently the characteristic equation (13) can be factorized into two biquadratic equations. The stiffness coefficients  $A_{j6}$ ,  $B_{j6}$  and  $D_{j6}$  represent coupling interaction while  $A_{45}$  depends predominantly on the shear moduli  $G_{31}$  and  $G_{23}$ . For a composite material where these shear moduli are approximately the same,  $A_{45}$  can be neglected.

The uncoupled form of the characteristic equation is

$$\begin{aligned} & [(F_{11}F_{22} - F_{21}^2)s^4 - (F_{11}A_{44}^0 + F_{22}A_{44}^1)s^2 + A_{44}^1A_{44}^0] \\ & [(F_{33}F_{44} - F_{43}^2)s^4 - (F_{33}A_{55}^0 + F_{44}A_{55}^1)s^2 + A_{55}^0A_{55}^1] = 0 \end{aligned} \quad (15)$$

The first bracket in Equation (15) control  $V$  and  $\beta_y$  behavior while the second  $U$  and  $\beta_x$ . The absolute values of the roots in Equation (15) will be denoted by  $s_{y1}$ ,  $s_{y2}$ ,  $s_{x1}$  and  $s_{x2}$ . These can be regarded as a good approximation for the roots of the coupled equation (13). A comparison between the coupled and uncoupled roots for a typical laminate is provided in Table III. These roots are found to be real for the material system and layup used. The predictions of the uncoupled equation (15) are in good approximation with the coupled equation (13). Also, the larger characteristic roots  $s_{x1}$  and  $s_{y1}$  correspond to bending behavior while the smaller  $s_{x2}$  and  $s_{y2}$  control the membrane behavior. This is shown in the following section.

Membrane behavior can be modeled by setting

$$M_x = M_y = M_{xy} = 0 \quad (16)$$

in the equilibrium equations for sublaminae 1 and 0. The characteristic equation (13) reduces to

$$(F_{11m}s^2 - A_{44}^1)(F_{22m}s^2 - A_{55}^1) = 0 \quad (17)$$

The membrane parameters  $F_{11m}$  and  $F_{22m}$  are defined in Equation (II-9) of Appendix II. They depend on  $A_{ij}$  and  $B_{ij}$  coefficients. The characteristic roots predicted by Equation (17) are included in Table III. By comparison with the roots of Equation (15), bending behavior is more localized than the membrane behavior as the characteristic roots controlling bending are larger. This fact is expected since Classical Lamination Theory (CLT), which predicts membrane-type behavior, prevails in the interior of the laminate.

Since the laminate width is large compared to its thickness and to the crack length, the response in sublaminates 1 and 0 is predominantly decaying from the crack tip and, therefore, only the roots with a negative sign will be considered in this solution.

Cracked Region of the Laminate:

Sublaminate 2

This sublaminate represents the upper group of plies in the cracked portion of the laminate. Since there is no restriction on the transverse displacement  $W$ , boundary conditions on  $Q$  can be specified.

The upper and lower surfaces in this sublaminate are stress-free and at  $y_1 = -a$  there is a free edge. The equilibrium equation (7) reduces to

$$N_{xy_2} = N_{y_2} = Q_{y_2} = M_{y_2} = 0$$

$$M_{xy_2,y} - Q_{x_2} = 0$$

(18)

By substituting these conditions into the constitutive relations (6), to a single differential equation in terms of  $\beta_{2x}$  obtained.

$$(D_{66}^1 + B_{26}^1 Cd_{12} + B_{66}^1 Cd_{22} + D_{26}^1 Cd_{32}) \beta_{2x,yy} - \left[ A_{55}^1 - \frac{(A_{45}^1)^2}{A_{44}^1} \right] \beta_{2x} = 0$$

(19)

Parameters  $Cd_{12}$ ,  $Cd_{22}$  and  $Cd_{32}$  are defined in Equation (II-10) of Appendix

II. Solution of Equation (18) leads to

$$\beta_{2x} = H_1 e^{s_c y_1} + H_2 e^{-s_c y_1}$$

where  $0 \leq y_1 \leq -a$

$$s_c = \left[ \frac{A_{55}^1 - (A_{45}^1)^2 / A_{44}^1}{(D_{66}^1 + B_{26}^1 C d_{12} + B_{66}^1 C d_{22} + D_{26}^1 C d_{32})} \right]^{1/2} \quad (20)$$

Since the crack length 'a' can be small, positive as well as negative signs of the root  $s_c$  have been considered. The arbitrary constants  $H_1$  and  $H_2$  are determined from the boundary condition at

$$y_1 = -a \quad M_{xy_2}(-a) = 0 \quad (21)$$

and continuity conditions at  $y_1 = 0$  with sublamine 1.

The displacements at the bottom surface of sublamine 2 are

$$\begin{aligned} v_2(y_1, -h_1/2) &= v_i(0) + (C d_{11} - \frac{h_1}{2} C d_{31}) \epsilon y_1 - (C d_{12} - \frac{h_1}{2} C d_{32}) \\ &\quad [H_1 (1 - e^{s_c y_1}) + H_2 (1 - e^{-s_c y_1})] \\ u_2(y_1 - h_1/2) &= u_i(0) + C d_{21} \epsilon y_1 - (C d_{22} - \frac{h_1}{2}) [H_1 (1 - e^{s_c y_1}) + H_2 (1 - e^{-s_c y_1})] \end{aligned} \quad (22)$$

where

$$v_i(0) = v_2(0, h_1/2)$$

$$u_i(0) = u_2(0 - h_1/2)$$

The linear terms in  $y_1$  in Equation (22) represent the displacement when  $\beta_{2x} = 0$ .

(ii) Sublamine 3

The upper surface of this sublamine is stress-free, while the interlaminar shear stresses on the lower surface are zero from symmetry conditions. Moreover, there is a free edge at  $y_0 = -a$ . The equilibrium equation (7) reduces to

$$\begin{aligned} N_{xy_3} &= N_{y_3} = 0 \\ M_{xy_3,y} - Q_{x_3} &= 0 \\ M_{y_3,y} - Q_{y_3} &= 0 \end{aligned} \quad (23)$$

By substituting these conditions into the constitutive relations (6), two coupled differential equations in  $\beta_{3y}$  and  $\beta_{3x}$  is obtained

$$\begin{bmatrix} J_{22}L_{yy} - A_{44}^0 & J_{26}L_{yy} - A_{45}^0 \\ J_{26}L_{yy} - A_{45}^0 & J_{66}L_{yy} - A_{55}^0 \end{bmatrix} \begin{bmatrix} \beta_{3y} \\ \beta_{3x} \end{bmatrix} = 0 \quad (24)$$

Parameters  $J_{22}$ ,  $J_{26}$  and  $J_{66}$  in Equation (24) are defined in Equation (II-12) of Appendix II.

Solve the differential equations for  $\beta_{3y}$  and  $\beta_{3x}$ .

$$\begin{aligned} \beta_{3y} &= I_1 e^{ss_1 y_0} + I_2 e^{-ss_1 y_0} + I_3 e^{ss_2 y_0} + I_4 e^{-ss_2 y_0} \\ \beta_{3x} &= \eta_1 (I_1 e^{ss_1 y_0} + I_2 e^{-ss_1 y_0}) + \eta_2 (I_3 e^{ss_2 y_0} + I_4 e^{-ss_2 y_0}) \end{aligned}$$

where

$$-a \leq y_0 \leq 0$$



$$\eta_j = - (J_{22}^{ss_j^2} - A_{44}^0) / (J_{26}^{ss_j^2} - A_{45}^0) \quad (25)$$

The constants  $I_1$  through  $I_4$  are found from the boundary conditions at

$$y_0 = -a \quad M_{y_3}(-a) = M_{xy_3}(-a) = 0 \quad (26)$$

and continuity conditions at  $y_0 = 0$  with sublaminate 0.

The roots  $ss_1$  and  $ss_2$  in Equation (25) are found by solving the characteristic equation resulting from equation (24).

The displacements at the upper surface of sublaminate 3 are given by

$$\begin{aligned} v_3(y_0, h_0/2) = & v_i(o) + wd_{11} \epsilon y_0 + (wd_{12} + \frac{h_0}{2} + \eta_1 wd_{13}) [I_1 (e^{ss_1 y_0} - 1) \\ & + I_2 (e^{-ss_1 y_0} - 1)] + (wd_{12} + \frac{h_0}{2} + \eta_2 wd_{13}) [I_3 (e^{ss_2 y_0} - 1) \\ & + I_4 (e^{-ss_2 y_0} - 1)] \\ u_3(y_0, h_0/2) = & u_i(o) + wd_{21} \epsilon y_0 + [wd_{22} + \eta_1 (wd_{23} + \frac{h_0}{2})] [I_1 (e^{ss_1 y_0} - 1) \\ & + I_2 (e^{-ss_1 y_0} - 1)] + [wd_{22} + \eta_2 (wd_{23} + \frac{h_0}{2})] [I_3 (e^{ss_2 y_0} - 1) \\ & + I_4 (e^{-ss_2 y_0} - 1)] \end{aligned} \quad (27)$$

Parameters  $wd_{11}$ ,  $wd_{12}$ ,  $wd_{13}$ ,  $wd_{21}$ ,  $wd_{22}$ , and  $wd_{23}$  are defined in Equation (II-12) of Appendix II.

In order to determine the energy release rate components by the virtual crack-closure method<sup>10</sup>, the relative displacements at the crack surface as well as the interlaminar stresses at the crack tip are needed.

From Equations (22) and (27) the relative displacements are

$$\begin{aligned}
\Delta v &= v_3(y_0, h_0/2) - v_2(y_1, -h_1/2) \\
&= (wd_{11} - Cd_{11} + \frac{h_1}{2} Cd_{31}) y\epsilon + (wd_{12} + \frac{h_0}{2} + \eta_1 wd_{13}) \\
&\quad [I_1 (e^{ss_1 y} - 1) + I_2 (e^{-ss_1 y} - 1)] \\
&\quad + (wd_{12} + \frac{h_0}{2} + \eta_2 wd_{13}) [I_3 (e^{ss_2 y} - 1) + I_4 (e^{-ss_2 y} - 1)] \\
&\quad + (Cd_{12} - \frac{h_1}{2} Cd_{32}) [H_1 (1 - e^{s_c y}) + H_2 (1 - e^{-s_c y})] \\
\Delta u &= u_3(y_0, h_0/2) - u_2(y_1, -h_1/2) \\
&= (wd_{21} - Cd_{21}) y\epsilon + [wd_{22} + \eta_1 (wd_{23} + \frac{h_0}{2})] [I_1 (e^{ss_1 y} - 1) \\
&\quad + I_2 (e^{-ss_1 y} - 1)] \\
&\quad + [wd_{22} + \eta_2 (wd_{23} + \frac{h_0}{2})] [I_3 (e^{ss_2 y} - 1) + I_4 (e^{-ss_2 y} - 1)] \\
&\quad + (Cd_{22} - \frac{h_1}{2}) [H_1 (1 - e^{s_c y}) + H_2 (1 - e^{-s_c y})] \\
&\qquad\qquad\qquad -a \leq y \leq 0 \\
&\qquad\qquad\qquad (28)
\end{aligned}$$

The linear terms in Equation (28) represent the relative displacements when the shear deformations  $\beta_{2x}$ ,  $\beta_{3x}$  and  $\beta_{3y}$  are neglected. The remaining terms are exponential, and their effect on the predictions are depicted in Figures 4 and 5, for two typical laminates. The dotted lines denoted by  $\Delta U_L$  and  $\Delta V_L$  in the figures correspond to the linear contribution, while the solid lines  $\Delta U$  and  $\Delta V$  include the exponential terms. The crack length 'a' in figures is 10 percent of the laminate semi-width and the applied strain is 1000 micro in/in. The simple linear distribution represents a good

approximation for the test cases considered. Due to the simplicity and accuracy of the linear displacement distribution, the energy release rate computations will be based on the linear displacement contribution only. A discussion of the effect of the exponential terms on the calculation of the total energy release rate will be provided later.

### Interlaminar Stresses

Denote the absolute values of the roots in Equation (13) by  $s_j$  ( $j=1,4$ ). The response can be written in the form

$$\beta_{1y} = G_j e^{-s_j y}$$

$$\beta_{1x} = G_j \alpha_j e^{-s_j y}$$

$$\beta_{oy} = G_j \psi_j e^{-s_j y}$$

$$\beta_{ox} = G_j \gamma_j e^{-s_j y}$$

$$U_{1,y} = C_v - s_j v_j G_j e^{-s_j y}$$

$$V_{1,y} = C_u - s_j u_j G_j e^{-s_j y}$$

$$j = 1,4; 0 \leq y \leq b$$

(29)

Summation over the range of index  $j$  is implied in Equation (29). Parameters  $\alpha_j$ ,  $\psi_j$ ,  $\gamma_j$ ,  $v_j$ , and  $u_j$  are defined in Equation (II-14) of Appendix II. The arbitrary constants of integration associated with the displacements are  $C_v$  and  $C_u$ .

Substitute from Equation (29) into the constitutive relationship for sublaminates 1 and 0 to get

$$\begin{bmatrix} N_{x_1} \\ N_{y_1} \\ N_{xy_1} \\ M_{y_1} \\ M_{xy_1} \end{bmatrix} = \begin{bmatrix} A_{11}^1 & A_{12}^1 & A_{16}^1 \\ A_{12}^1 & A_{22}^1 & A_{26}^1 \\ A_{16}^1 & A_{26}^1 & A_{66}^1 \\ B_{12}^1 & B_{22}^1 & B_{26}^1 \\ B_{16}^1 & B_{26}^1 & B_{66}^1 \end{bmatrix} \begin{bmatrix} \epsilon \\ C_v \\ C_u \end{bmatrix} - \begin{bmatrix} N_{x_{1j}} \\ N_{y_{1j}} \\ N_{xy_{1j}} \\ M_{y_{1j}} \\ M_{xy_{1j}} \end{bmatrix} s_j G_j e^{-s_j y}$$

$$\begin{bmatrix} Q_{y_1} \\ Q_{x_1} \end{bmatrix} = \begin{bmatrix} A_{44}^1 & A_{45}^1 \\ A_{45}^1 & A_{55}^1 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_j \end{bmatrix} G_j e^{-s_j y}$$

$$\begin{bmatrix} N_{x_0} \\ N_{y_0} \\ N_{xy_0} \\ M_{y_0} \\ M_{xy_0} \end{bmatrix} = \begin{bmatrix} A_{11}^0 & A_{12}^0 & A_{16}^0 \\ A_{12}^0 & A_{22}^0 & A_{26}^0 \\ A_{16}^0 & A_{26}^0 & A_{66}^0 \\ B_{12}^0 & B_{22}^0 & B_{26}^0 \\ B_{16}^0 & B_{26}^0 & B_{66}^0 \end{bmatrix} \begin{bmatrix} \epsilon \\ C_v \\ C_u \end{bmatrix} - \begin{bmatrix} N_{x_{0j}} \\ -N_{y_{1j}} \\ -N_{xy_{1j}} \\ M_{y_{0j}} \\ M_{xy_{0j}} \end{bmatrix} s_j G_j e^{-s_j y}$$

$$\begin{bmatrix} Q_{y_0} \\ Q_{x_0} \end{bmatrix} = \begin{bmatrix} A_{44}^0 & A_{45}^0 \\ A_{45}^0 & A_{55}^0 \end{bmatrix} \begin{bmatrix} \psi_j \\ \gamma_j \end{bmatrix} G_j e^{-s_j y}$$

(30)

The parameters associated with Equation (30) are defined in Equation (II-15) of Appendix II.

There are twelve arbitrary constants of integration namely:  $G_j$  ( $j=1,4$ ),  $C_v$ ,  $C_u$ ,  $H_1$ ,  $H_2$  and  $I_1$  through  $I_4$ . These can be found from the boundary condition at the free edge of sublaminates 2 and 3 expressed in Equations (21) and (26) and from the continuity conditions between sublaminates 1 and 2, and 0 and 3 at  $y = 0$ . These are:

$$N_{y_1}(0) = N_{y_2}(0) = 0$$

$$N_{xy_1}(0) = N_{xy_2}(0) = 0$$

$$M_{y_1}(0) = M_{y_2}(0) = 0$$

$$N_{y_0}(0) = N_{y_3}(0) = 0$$

$$N_{xy_0}(0) = N_{xy_3}(0) = 0$$

$$M_{xy_2}(0) = M_{xy_1}(0)$$

$$M_{xy_0}(0) = M_{xy_3}(0)$$

$$M_{y_0}(0) = M_{y_3}(0)$$

$$\beta_{2x}(0) = \beta_{1x}(0)$$

$$\beta_{0x}(0) = \beta_{3x}(0)$$

$$\beta_{0y}(0) = \beta_{3y}(0) \quad (31)$$

Boundary conditions on  $Q_y$  cannot be specified in sublaminate 1 and 0 as a result of neglecting the transverse normal strain in the assumed displacement function. Equations (21), (26) and (31) represent 14 boundary conditions for twelve constants. However, the last two continuity conditions at the interface between sublaminate 0 and 3 in Equation (31) lead to

$$A_{44}^0 \beta_{oy}(0) + A_{45}^0 \beta_{ox}(0) = A_{44}^0 \beta_{3y}(0) + A_{45}^0 \beta_{3x}(0)$$

that is,

$$Q_{y_0}(0) = Q_{y_3}(0) \quad (32)$$

which cannot be specified. Therefore, the conditions  $\beta_{ox}(0) = \beta_{3x}(0)$  and  $\beta_{oy}(0) = \beta_{3y}(0)$  cannot be prescribed for consistency.

Substitute for the resultant forces and moments in sublaminate 0 through 3 into the first nine equations in (31) and solve for the arbitrary constants. As  $C_v$  and  $C_u$  are easily obtained, their expressions are listed below for convenience.

$$C_v = (k_{26} k_{16} - k_{66} k_{12}) \varepsilon / D$$

$$C_u = (k_{26} k_{12} - k_{22} k_{16}) \varepsilon / D$$

where

$$\begin{aligned} k_{12} &= A_{12}^1 + A_{12}^0 \\ k_{16} &= A_{16}^1 + A_{16}^0 \end{aligned} \quad (33)$$

Parameters  $k_{26}$ ,  $k_{66}$ ,  $k_{22}$  and  $D$  are defined in Equations (II-3) and (II-4).

The resultant axial forces and moments in Equation (30) consist of two parts: a constant term and an exponentially decaying term. In the interior of the laminate the second term is negligible and the response is controlled by the first term. Since CLT is retrieved in the interior of the laminate, the first term can be recognized as the CLT prediction. This is shown in Appendix III where a CLT solution is derived for the ED specimen. This solution is simple and can be derived on a ply-by-ply basis. Moreover, the sign of the resultant axial force  $N_y$  determines the sign of the interlaminar peel stress. A compressive peel stress tend to retard delamination at a given interface.

Interlaminar stresses at the interface between sublaminae 1 and 0 are found from equilibrium.

$$\begin{aligned}
 t_x &= N_{xy_1,y} = N_{x_1j} s_j^2 G_j e^{-s_j y} \\
 t_y &= N_{y_1,y} = N_{y_1j} s_j^2 G_j e^{-s_j y} \\
 p &= Q_{y_1,y} = -s_j (A_{44}^{-1} + A_{45}^{-1} \alpha_j) G_j e^{-s_j y}
 \end{aligned} \tag{34}$$

The distribution of the peel stress  $p$  is not in equilibrium since the boundary condition on the shear force  $Q_{y_1}(0)$  cannot be prescribed.

$$\int_0^b p dy = \int_0^b Q_{y_1,y} dy = Q_{y_1}(b) - Q_{y_1}(0) \tag{35}$$

$Q_{y_1}(b) \approx 0$  as  $Q_y$  is an exponentially decaying function, however  $Q_{y_1}(0) \neq 0$ .

A comparison of the interlaminar shear stress distribution  $t_y$  along the interface between sublaminates 1 and 0 appears in Figure 6 for three laminates. The applied uniform strain is 1000 micro in/in.

### Energy Release Rate

The total energy release rate can be determined by considering the work done by external forces<sup>5</sup>. For a uniform applied strain  $\epsilon_x = \epsilon$  throughout the laminate this can be written as

$$G_T = \frac{-\epsilon}{2} \frac{dP}{da} \quad (36)$$

where P is the axial applied force and is given by

$$P = \int_0^{(b-a)} (N_{x_1} + N_{x_0}) dy + \int_{-a}^0 (N_{x_2} + N_{x_3}) dy \quad (37)$$

The axial forces  $N_{x_1}$  and  $N_{x_0}$  in sublaminate 1 and 0 respectively are given in Equation (30) while the axial forces in sublaminate 2 and 3 are given by

$$\begin{aligned} N_{x_2} &= (A_{11}^1 + A_{12}^1 Cd_{11} + A_{16}^1 Cd_{21} + B_{12}^1 Cd_{31})\epsilon + J_1 s_c (H_1 e^{s_c y} - H_2 e^{-s_c y}) \\ N_{x_3} &= (A_{11}^0 + A_{12}^0 wd_{11} + A_{16}^0 wd_{21})\epsilon + (J_2 + \eta_1 J_3) ss_1 (I_1 e^{ss_1 y} - I_2 e^{-ss_1 y}) \\ &\quad + (J_2 + \eta_2 J_3) ss_2 (I_3 e^{ss_2 y} - I_4 e^{-ss_2 y}) \end{aligned} \quad (38)$$



where

$$\begin{aligned}
 J_1 &= B_{16}^1 + A_{12}^1 Cd_{12} + A_{16}^1 Cd_{22} + B_{12}^1 Cd_{32} \\
 J_2 &= B_{12}^0 + A_{12}^0 wd_{12} + A_{16}^0 wd_{22} \\
 J_3 &= B_{16}^0 + A_{13}^0 wd_{13} + A_{16}^0 wd_{23}
 \end{aligned} \tag{39}$$

Substitute from Equations (30) and (38) into Equation (36) and using Equation (37) to get

$$\begin{aligned}
 G_T = \frac{\epsilon}{2} [K_{12} C_v + K_{16} C_u - (A_{12}^1 Cd_{11} + A_{16}^1 Cd_{21} + B_{12}^1 Cd_{31})\epsilon \\
 - (A_{12}^0 wd_{11} + A_{16}^0 wd_{21})\epsilon] + G_R
 \end{aligned} \tag{40}$$

where  $G_R$  is an exponential function of the crack length. For a crack length larger than a few ply thicknesses, the contribution of  $G_R$  is negligible. This situation is depicted in Figure 7 where a normalized total energy release rate for a typical laminate is plotted against the crack length divided by ply thickness. The laminate thickness is denoted by  $t$  in the figure and the ply thickness is  $h$ . In this case,  $G_T$  reaches the constant value predicted by the first term in Equation (40) for a value of crack length larger than four ply thickness. Also appearing in Figure 7 is a comparison with a finite element solution presented in Figure 6 of Reference 4.

The total energy release rate is a global parameter which does not depend strongly on the local details at the crack tip. This is the reason why relatively simple modeling approaches yield adequate predictions for the total energy release rate.

This finding checks the results of References 4 and 8 where simple closed-form expressions for  $G_I$  are in good agreement with predictions based on refined theories or on finite element simulation.

#### Energy Release Rate Components

Using the virtual crack-closure method  $G_{II}$  and  $G_{III}$  are given by

$$G_{II} = \lim_{\delta \rightarrow 0} \frac{1}{2\delta} \int_0^\delta t_y(\delta-r) \Delta v(r) dr$$

$$G_{III} = \lim_{\delta \rightarrow 0} \frac{1}{2\delta} \int_0^\delta t_x(\delta-r) \Delta u(r) dr$$

(41)

where  $\delta$  is a virtual crack step size. Unless a singularity exists in the stress field, Equation (41) yields the trivial result ( $G_{II} = G_{III} = 0$ ) when the limit as  $\delta$  tends to zero is determined.<sup>9</sup> Consequently, a sufficiently large finite step size is essential to get an answer when using models that do not exhibit singular behavior.

Substitute from Equation (34) for the interlaminar stresses in term of force resultants into Equation (41) and use the linear terms in Equation (28) for the relative displacements to get

$$G'_{II} = \frac{1}{2} \phi_v \bar{F}_y$$

$$G'_{III} = \frac{1}{2} \phi_u \bar{F}_x$$

where

$$\theta_v = w d_{11} - C d_{11} + \frac{h_1}{2} C d_{31}$$

$$\theta_u = w d_{21} - C d_{21}$$

$$\bar{F}_y = \frac{1}{\Delta} \int_0^{\Delta} N_{y1} dy$$

$$\bar{F}_x = \frac{1}{\Delta} \int_0^{\Delta} N_{xy1} dy$$

(42)

The finite crack step size is denoted by  $\Delta$ . The resultant forces  $N_{y1}$  and  $N_{xy1}$  exhibit a boundary layer behavior. This is shown in Figures 8 and 9 where the resultant force distributions are plotted along the laminate width. In the interior a constant value corresponding to the CLT prediction is reached.

The average resultant force values  $\bar{F}_y$  and  $\bar{F}_x$  depends on the selected value of the crack step  $\Delta$ . Recommended ad hoc values as a percentage of the initial crack length have been suggested in the finite element representation of the crack-closure method.<sup>11</sup> However, within the boundary layer region  $N_{y1}$  and  $N_{xy1}$  have steep gradients and consequently a small variation in the selection of  $\Delta$  leads to large variations in  $\bar{F}_y$  and  $\bar{F}_x$ . Since  $N_{y1}$  and  $N_{xy1}$  distributions at the crack tip are controlled by the boundary layer decay length, the crack step size  $\Delta$  should be selected based on the boundary layer length rather than a percentage of the initial crack length.

Mode I energy release rate is found from

$$G_I = G_T - (G_{II} + G_{III}) \quad (43)$$

$G_{II}$  and  $G_{III}$  are given in Equation (42) and  $G_T$  in Equation (40).

The energy release rate components predicted by this approach show a good correlation with finite element solutions when the crack step size  $\Delta$  is selected as

$$\frac{\Delta}{b} = [\bar{s}_4 (\bar{s}_1 + \bar{s}_2 + \bar{s}_3)]^2 \quad (44)$$

where  $\bar{s}_j$  ( $j=1,4$ ) are the characteristic roots in increasing magnitude nondimensionalized by the ply thickness  $h$ . The distinct roots  $s_j$  control the decay length associated with different physical variables. It has been found empirically that the combination above gives very good correlation with finite element simulations for over sixty cases that have been compared. Although the functional form is not simply identified with a boundary length it nevertheless contains the proper information.

For a general layup the crack size  $\Delta$  is influenced by all four roots. However, for laminates where Mode III is negligible the crack step size is influenced by the characteristic roots  $s_2$  and  $s_4$  that control  $V$  and  $\beta_y$  behavior. For this situation, the following crack step size expression provides good correlation with the finite element solution:

$$\frac{\Delta}{b} = 2.6 (\bar{s}_4 \bar{s}_2)^2 \quad (45)$$

### Results and Discussion

An extensive comparison between the energy release rate components predicted by the present approach and a quasi 3-D finite element solution has been performed. The results appears in Tables IV through X and in Figures 10 through 28. The finite element results in Tables V through X

are related to the work of Reference 4 and were provided by the authors. The properties and geometry used appear in Table I. The applied uniform strain is one micro in/in.

In Table IV a comparison is provided for six laminates. The first three layups were reported in Reference 3. The  $G_{III}$  component in these layups are negligible. The remaining layups represent cases where  $G_I$ ,  $G_{II}$  and  $G_{III}$  components are all finite.

The laminates presented in Tables V-X are rotated stacking sequences for two quasi-isotropic layups. The first is a  $[0/90/+45]_S$  and the second is a  $[45/90/-45/0]_S$ . The results of the first laminate appear in Tables V-VII while in Tables VIII-X the results of the second laminate are provided. These laminates were used in Reference 4 to investigate delaminations around an open hole. The strain energy release rate distribution around the hole boundary, for delaminations growing in a prescribed interface, was calculated by assuming that each circumferential portion acts as a straight edge subjected to an appropriate uniform circumferential strain. Hence, at each circumferential angle  $\theta$  the laminate reflects a new stacking sequence where the load is applied in the  $\theta$ -direction tangent to the hole. The angular position corresponding to each stacking sequence is provided in Tables V-X.

The results in Table V are plotted in Figures 10-12. In Figures 10 and 11 the percentages of  $G_{II}$  and  $G_{III}$  are plotted against the angular position,  $\theta$ , around a hole in the first laminate. The finite element results are shown in solid lines while the results of the present analysis appear in dotted lines. A comparison of  $G_T$  is shown in Figure 12. Similarly, the results in Table VI and VII are plotted in Figures 13-16 and

17-19, respectively. The results of the second laminate are provided in Tables VIII-X. The results in Table VIII are shown in Figures 20-22. Similarly the results of Tables IX and X are plotted in Figures 23-25 and 26-28. In Figures 22, 25 and 28 as well as in Tables VIII-X the values of  $G_T$  from the simple expression derived by O'Brien<sup>4</sup> are included for comparison. The plots in Figures 26-28 have been discontinued at  $\theta = -45^\circ$  as compressive peel stress occurs at the 45/90 interface. The sign of the peel stress at a given interface can be determined from a simple membrane-type model. This is shown in Appendix III. Under compressive peel stress the crack surfaces tend to close and a special modeling approach should be used. One possible approach was proposed in Reference 9 in connection with a double cracked-lap-shear specimen tested under compression loading.

### Conclusion

Interlaminar stresses and energy release rates are estimated for the ED test using a shear-type deformation theory and a sublaminar approach. The predictions are obtained in closed form and the parameters controlling the behavior are identified. The governing equations are derived using a virtual work approach. Due to the absence of transverse strain the interlaminar peel stress distribution is not in equilibrium. The interlaminar shear stresses, however, show reliable trends. The energy release rate components  $G_{II}$  and  $G_{III}$  are estimated based on the interlaminar shear stresses and relative displacements using the virtual crack closure method. The total energy release,  $G_T$ , is determined from the

rate of change of the work done by the external forces with crack length. Then  $G_I$  is obtained as the difference between  $G_T$  and  $(G_{II} + G_{III})$ .

An extensive comparison between the energy release rates predicted by the present approach and a quasi 3-D finite element solution for over sixty test cases, is performed. The agreement is good. The methodology outlined in this work is simple and the results are generated using a Hewlett Packard 9845B desktop computer.

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## Appendix I

### Derivation of the Governing Equations

In this Appendix the governing equations for the sublaminate shown in Figure 3 are derived using the principle of virtual work.

Consider a sublaminate... a single ply or group of plies conveniently treated as laminated units of thickness  $h$ . The origin of a cartesian coordinate system is located within the central plane ( $x$ - $y$ ) with the  $z$ -axis being normal to this plane. The material of each ply is assumed to possess a plane of elastic symmetry parallel to  $xy$  as shown in Figure 3.

Stress and moment resultants,

$$\begin{aligned} (N_x, N_y, N_{xy}, Q_x, Q_y) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xz}, \tau_{yz}) dz \\ (M_x, M_y, M_{xy}) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) z dz \end{aligned} \quad (I-1)$$

Because of the existence of a plane of elastic symmetry, the constitutive relations are given by

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & & & \\ C_{12} & C_{22} & & \text{SYM} \\ C_{13} & C_{23} & C_{33} & \\ C_{16} & C_{26} & C_{36} & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \tau_{yz} \\ \tau_{xz} \end{bmatrix} = \begin{bmatrix} C_{44} & \text{SYM} \\ C_{45} & C_{55} \end{bmatrix} \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}$$

(I-2)



where  $C_{ij}$  are components of the anisotropic stiffness matrix and  $\gamma_{xy}$ ,  $\gamma_{yz}$  and  $\gamma_{xz}$  are engineering shear strains

The displacements are assumed to be of the form

$$\begin{aligned} u &= U(x,y) + z\beta_x(y) \\ v &= V(y) + z\beta_y(y) \\ w &= W(y) \end{aligned} \quad (I-3)$$

where  $u, v$  and  $w$  are the displacement components in the  $x, y$  and  $z$  directions, respectively. Equation (I-3) in conjunction with the strain-displacement relations of classical theory of elasticity leads to the following kinematic relations

$$\begin{aligned} \epsilon_{xx} &= U_{,x} \\ \epsilon_{yy} &= V_{,y} + z\beta_{y,y} \\ \epsilon_{zz} &= 0 \\ \gamma_{xy} &= V_{,y} + z\beta_{x,y} \\ \gamma_{xz} &= \beta_x \\ \gamma_{yz} &= \beta_y + W_{,y} \end{aligned} \quad (I-4)$$

Substitute Equation (I-4) into Equation (I-2) and put the results into Equation (I-1). This yields the following constitutive relations:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} \\ 0 \\ \beta_{y,y} \\ \beta_{x,y} \end{bmatrix}$$

$$\begin{bmatrix} Q_y \\ Q_x \end{bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{bmatrix} \beta_y + w_{,y} \\ \beta_x \end{bmatrix}$$

where

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} C_{ij} (1, z, z^2) dz \quad (I-5)$$

The usual coupling between bending and extensional modes for a laminate of arbitrary construction occurs in Equation (I-5) through the stiffness terms  $B_{ij}$ .

The variation of the strain energy due to virtual displacements  $\delta u$ ,  $\delta v$  and  $\delta w$  is

$$\delta \bar{V} = \int_V (\sigma_x \delta \epsilon_x + \sigma_y \delta \epsilon_y + \sigma_z \delta \epsilon_z + \tau_{xy} \delta \gamma_{xy} + \tau_{yz} \delta \gamma_{yz} + \tau_{xz} \delta \gamma_{xz}) dV \quad (I-6)$$

where  $\delta\epsilon_x$ ,  $\delta\epsilon_y$ ,  $\delta\epsilon_z$ ,  $\delta\gamma_{xy}$ ,  $\delta\gamma_{yz}$  and  $\delta\gamma_{xz}$  are the strains associated with the virtual displacements. Substitute from Equation (I-3) and integrate through the thickness using Equation (I-1) to get

$$\delta\bar{U} = \int_A [N_x \delta U_{,x} + N_y \delta V_{,y} + N_{xy} \delta U_{,y} + Q_x \delta\beta_x + Q_y (\delta\beta_y + \delta W_{,y}) + M_x \delta\beta_{x,x} + M_y \delta\beta_{y,y} + M_{xy} \delta\beta_{x,y}] dA \quad (I-7)$$

The variation of the work done by the external forces and by the surface fractions is

$$\delta\bar{W} = \int_A (n_x \delta U + n_y \delta V + q \delta W + m_x \delta\beta_x + m_y \delta\beta_y) dA + \int_S (\bar{N}_n \delta\bar{U}_n + \bar{N}_{ns} \delta\bar{U}_s + \bar{M}_n \delta\bar{\beta}_n + \bar{M}_{ns} \delta\bar{\beta}_s) ds \quad (I-8)$$

where a bar denotes values on the boundary, n and s are coordinates normal and tangential to the edge, and

$$\begin{aligned} n_x &= t_{2x} - t_{1x} \\ n_y &= t_{2y} - t_{1y} \\ q &= p_2 - p_1 \\ m_x &= \frac{h}{2} (t_{2x} + t_{1x}) \\ m_y &= \frac{h}{2} (t_{2y} + t_{1y}) \end{aligned} \quad (I-9)$$

where  $n_x$  and  $n_y$  can be regarded as effective distributed axial forces,  $m_x$  and  $m_y$  effective distributed moments and q an effective lateral pressure.

From the principle of virtual work the equations of equilibrium and boundary conditions are determined from the Euler equations and boundary conditions of the variational equation.

$$\delta \bar{V} = \delta \bar{W} \quad (I-10)$$

Substitution of Equations (I-7) and (I-8) into Equation (I-10) leads the following equations of equilibrium:

$$\begin{aligned} N_{x,x} + N_{xy,y} + n_x &= 0 \\ N_{xy,x} + N_{y,y} + n_y &= 0 \\ Q_{x,x} + Q_{y,y} + q &= 0 \\ M_{x,x} + M_{xy,y} - Q_x + m_x &= 0 \\ M_{xy,x} + M_{y,y} - Q_y + m_y &= 0 \end{aligned} \quad (I-11)$$

and one member of the following five products must be prescribed on the sublaminar edges

$$N_n U_n, N_{ns} U_s, M_n \beta_n, M_{ns} \beta_s \text{ and } Q_n W. \quad (I-12)$$

For the ED specimen under uniform extension,  $U(x,y)$  in Equation (I-3) is given by

$$U(x,y) = x\epsilon + U^*(y) \quad (I-13)$$

and the response is a function of  $y$  and  $z$  coordinates only. For this case the equilibrium equations (I-11) take the form

$$N_{xy,y} + n_x = 0$$

$$N_{y,y} + n_y = 0$$

$$Q_{y,y} + q = 0$$

$$M_{xy,y} - Q_x + m_x = 0$$

$$M_{y,y} - Q_y + m_y = 0$$

(I-14)

Substitution of the constitutive relations in Equation (I-5) into Equation

$$\begin{bmatrix} A_{66}L_{yy} & & & & & \\ A_{26}L_{yy} & A_{22}L_{yy} & & & & \\ 0 & 0 & -A_{44}L_{yy} & & & \\ B_{66}L_{yy} & B_{26}L_{yy} & -A_{45}L_y & (D_{66}L_{yy} - A_{55}) & & \\ B_{26}L_{yy} & B_{22}L_{yy} & -A_{44}L_y & (D_{26}L_{yy} - A_{45}) & (D_{22}L_{yy} - A_{44}) & \end{bmatrix} \begin{matrix} \text{SYM} \\ \\ \\ \\ \end{matrix} \begin{bmatrix} U^* \\ V \\ W \\ \beta_x \\ \beta_y \end{bmatrix} = - \begin{bmatrix} n_x \\ n_y \\ -q \\ m_x \\ m_y \end{bmatrix}$$

(I-15)

where the operators

$$L_{yy} = d^2/dy^2$$

$$L_y = d/dy$$

(I-16)

and for the boundary conditions at  $y = \text{constant}$  prescribe

$$N_{xy} \text{ or } U^*, N_y \text{ or } V, Q_y \text{ or } W, M_y \text{ or } \beta_y \text{ and } M_{xy} \text{ or } \beta_x \quad (I-17)$$

## Appendix II

### Definition of Parameters

In this appendix the parameters in Equations (11) - (30) are defined in terms of the sublaminate thickness and  $A_{ij}$ ,  $B_{ij}$  and  $D_{ij}$  coefficients.

Sublaminate 1 and 0:

The  $F_{ij}$  parameters ( $i=1-3$ ,  $j=1-4$ ) in Equation (11) are defined as

$$F_{11} = D_{22}^1 + \frac{h_1}{2} B_{22}^1 + h_{22} v_{11} + h_{26} u_{11}$$

$$F_{21} = h_{22} v_{12} - h_{26} u_{12}$$

$$F_{31} = D_{26}^1 + \frac{h_1}{2} B_{26}^1 + h_{22} v_{13} + h_{26} u_{13}$$

$$F_{41} = h_{22} v_{14} + h_{26} u_{14}$$

$$F_{22} = D_{22}^0 - \frac{h_0}{2} B_{22}^0 + C_{22} v_{12} + C_{26} u_{12}$$

$$F_{32} = h_{26} v_{12} + h_{66} u_{12}$$

$$F_{43} = h_{26} v_{14} + h_{66} u_{14}$$

$$F_{33} = D_{66}^1 + \frac{h_1}{2} B_{66}^1 + h_{26} v_{13} + h_{66} u_{13}$$

$$F_{42} = D_{26}^0 - \frac{h_0}{2} B_{26}^0 + C_{22} v_{14} + C_{26} u_{14}$$

$$F_{44} = D_{66}^0 - \frac{h_0}{2} B_{66}^0 + C_{26} v_{14} + C_{66} u_{14} \quad (\text{II-1})$$

Parameters  $u_{1i}$  and  $v_{1i}$  ( $i=1-4$ ) in Equation (II-1) relate the displacements  $U_1$  and  $V_1$  to the rotations through the following equation

$$\begin{bmatrix} v_{1,yy} \\ u_{1,yy} \end{bmatrix} = \begin{bmatrix} v_{11} L_{yy} & v_{12} L_{yy} & v_{13} L_{yy} & v_{14} L_{yy} \\ u_{11} L_{yy} & u_{12} L_{yy} & u_{13} L_{yy} & u_{14} L_{yy} \end{bmatrix} \begin{bmatrix} \beta_{1y} \\ \beta_{oy} \\ \beta_{1x} \\ \beta_{ox} \end{bmatrix}$$

(II-2)

where

$$v_{11} = (K_{26} h_{26} - K_{66} h_{22})/D + h_1/2$$

$$v_{12} = (K_{26} C_{26} - K_{66} C_{22})/D + h_o/2$$

$$v_{13} = (K_{26} h_{66} - K_{66} h_{26})/D$$

$$v_{14} = (K_{26} C_{66} - K_{66} C_{26})/D$$

$$u_{11} = (K_{26} h_{22} - K_{22} h_{26})/D$$

$$u_{12} = (K_{26} C_{22} - K_{22} C_{26})/D$$

$$u_{13} = (K_{26} h_{26} - K_{22} h_{66})/D + h_1/2$$

$$u_{14} = (K_{26} C_{26} - K_{22} C_{66})/D + h_o/2$$

$$D = K_{22} K_{66} - K_{26}^2$$

(II-3)

and

$$h_{22} = B_{22}^1 + \frac{h_1}{2} A_{22}^1$$

$$h_{26} = B_{26}^1 + \frac{h_1}{2} A_{26}^1$$

$$h_{66} = B_{66}^1 + \frac{h_1}{2} A_{66}^1$$

$$C_{22} = B_{22}^0 + \frac{h_0}{2} A_{22}^1$$

$$C_{26} = B_{26}^0 + \frac{h_0}{2} A_{26}^1$$

$$C_{66} = B_{66}^0 + \frac{h_0}{2} A_{66}^1$$

$$K_{22} = A_{22}^1 + A_{22}^0$$

$$K_{26} = A_{26}^1 + A_{26}^0$$

$$K_{66} = A_{66}^1 + A_{66}^0$$

(II-4)

The coefficients  $E_8$  through  $E_0$  in Equation (13) are defined as

$$E_8 = F_{11}W_1 - F_{31}X_1 + F_{21}Y_1 - F_{41}Z_1$$

$$E_6 = F_{11}W_2 - A_{44}^1 W_1 - F_{31}X_2 + A_{45}^1 X_1 + F_{21}Y_2 - F_{41}Z_2$$

$$E_4 = [A_{44}^0 A_{55}^0 - (A_{45}^0)^2] (F_{11}F_{33} - F_{31}^2) + (F_{22}A_{55}^0 + F_{44}A_{44}^0 - 2F_{42}A_{45}^0) (F_{11}A_{55}^1 - F_{31}A_{45}^1) - A_{44}^1 W_2 + A_{45}^1 X_2 + F_{21}Y_3 - F_{41}Z_3$$

$$E_2 = -[A_{44}^0 A_{55}^0 - (A_{45}^0)^2] [F_{11}A_{55}^1 + F_{33}A_{44}^1 - 2F_{31}A_{45}^1] - [A_{44}^1 A_{55}^1 - (A_{45}^1)^2] (F_{22}A_{55}^0 + F_{44}A_{44}^0 - 2F_{42}A_{45}^0)$$

$$E_0 = [A_{44}^0 A_{55}^0 - (A_{45}^0)^2] [A_{44}^1 A_{55}^1 - (A_{45}^1)^2]$$



where

$$W_1 = F_{33} (F_{22} F_{44} - F_{42}^2) - F_{32}^2 F_{44} + 2F_{43} F_{42} - F_{43}^2 F_{22}$$

$$W_2 = F_{33} (F_{22} A_{55}^0 + F_{44} A_{44}^0 - 2F_{42} A_{45}^0) - A_{55}^1 (F_{22} F_{44} - F_{42}^2) \\ + F_{32}^2 A_{55}^0 - 2F_{43} F_{32} A_{45}^0 + F_{43}^2 A_{44}^0$$

$$X_1 = F_{31} (F_{22} F_{44} - F_{42}^2) - F_{32} (F_{21} F_{44} - F_{41} F_{42}) + F_{43} (F_{21} F_{42} - F_{41} F_{22})$$

$$X_2 = -F_{31} (F_{22} A_{55}^0 + F_{44} A_{44}^0 - 2F_{42} A_{45}^0) - A_{45}^1 (F_{22} F_{44} - F_{42}^2) \\ + F_{32} (F_{21} A_{55}^0 - F_{41} A_{45}^0) - F_{43} (F_{21} A_{45}^0 - F_{41} A_{44}^0)$$

$$Y_1 = F_{31} (F_{32} F_{44} - F_{43} F_{42}) - F_{33} (F_{21} F_{44} - F_{41} F_{42}) + F_{43} (F_{21} F_{43} - F_{41} F_{32})$$

$$Y_2 = -A_{45}^1 (F_{32} F_{44} - F_{43} F_{42}) - F_{31} (F_{32} A_{55}^0 - F_{43} A_{45}^0) \\ + A_{55}^1 (F_{21} F_{44} - F_{41} F_{42}) + F_{33} (F_{21} A_{55}^0 - F_{41} A_{45}^0)$$

$$Y_3 = A_{45}^1 (F_{32} A_{55}^0 - F_{43} A_{45}^0) - A_{55}^1 (F_{21} A_{55}^0 - F_{41} A_{45}^0)$$

$$Z_1 = F_{31} (F_{32} F_{42} - F_{43} F_{22}) - F_{33} (F_{21} F_{42} - F_{41} F_{22}) + F_{32} (F_{21} F_{43} - F_{41} F_{32})$$

$$Z_2 = F_{31} (F_{43} A_{44}^0 - F_{32} A_{45}^0) - A_{45}^1 (F_{32} F_{42} - F_{43} F_{22}) + A_{55}^1 (F_{21} F_{42} - F_{41} F_{22}) \\ - F_{33} (F_{41} A_{44}^0 - F_{21} A_{45}^0)$$

$$Z_3 = -A_{45}^1 (F_{43} A_{44}^0 - F_{32} A_{45}^0) + A_{55}^1 (F_{41} A_{44}^0 - F_{21} A_{45}^0)$$

(II-5)

when the coefficients  $A_{j6}$ ,  $B_{j6}$ ,  $D_{j6}$  ( $j=1,2$ ) and  $A_{45}$  are neglected

we have

$$v_{13} = v_{14} = u_{11} = u_{12} = 0$$

and

$$F_{31} = F_{41} = F_{32} = F_{42} = 0 \quad (II-6)$$

for this case Equation (II-2) is

$$\begin{bmatrix} v_{1,yy} \\ u_{1,yy} \end{bmatrix} = \begin{bmatrix} v_{11}L_{yy} & v_{12}L_{yy} & 0 & 0 \\ 0 & 0 & u_{13}L_{yy} & u_{14}L_{yy} \end{bmatrix} \begin{bmatrix} \beta_{1y} \\ \beta_{oy} \\ \beta_{1x} \\ \beta_{ox} \end{bmatrix}$$

(II-7)

and Equation (11) takes the uncoupled form

$$\begin{bmatrix} (F_{11}L_{yy} - A_{44}^1) & & & \\ F_{21}L_{yy} & (F_{22}L_{yy} - A_{44}^0) & & \\ 0 & 0 & (F_{33}L_{yy} - A_{55}^1) & \\ 0 & 0 & F_{43}L_{yy} & (F_{44}L_{yy} - A_{55}^0) \end{bmatrix} \begin{bmatrix} \beta_{1y} \\ \beta_{oy} \\ \beta_{1x} \\ \beta_{ox} \end{bmatrix} = 0$$

SYM

(II-8)

By neglecting bending effects in sublaminates 1 and 0, the rotations  $\beta_{oy}$  and  $\beta_{ox}$  can be expressed in terms of  $\beta_{1y}$  and  $\beta_{1x}$  and the characteristic equation reduces to

$$(F_{11m}s^2 - A_{44}^1)(F_{22m}s^2 - A_{55}^1) = 0$$

where

$$F_{11m} = \frac{h_1}{2} [A_{22}^1 (v_{11} + v_{12} \frac{h_o}{h_1} \frac{A_{44}^1}{A_{44}^o}) + B_{22}^1]$$

$$F_{22m} = \frac{h_2}{2} [A_{66}^1 (u_{13} + u_{14} \frac{h_o}{h_1} \frac{A_{55}^1}{A_{55}^o}) + B_{66}^1]$$

(II-9)

Parameters  $F_{11m}$  and  $F_{22m}$  represent a membrane behavior and therefore do not depend on the bending coefficients  $D_{ij}$ .

#### Sublamine 2:

The elements of the matrix Cd in Equation (19) are defined as

$$[Cd]_{3 \times 2} = - \begin{bmatrix} A_{22}^1 & & & \text{SYM} \\ A_{26}^1 & A_{66}^1 & & \\ B_{22}^1 & B_{26}^1 & D_{22}^1 & \end{bmatrix}^{-1} \begin{bmatrix} A_{12}^1 & B_{26}^1 \\ A_{16}^1 & B_{66}^1 \\ B_{12}^1 & D_{26}^1 \end{bmatrix}$$

(II-10)

Displacements  $U_2$  and  $V_2$  and rotation  $\beta_{2y}$  in sublamine 2 are related to the applied uniform strain  $\epsilon$  and the rotation  $\beta_{2x}$  through the matrix Cd by

$$\begin{bmatrix} v_{2,y} \\ u_{2,y} \\ \beta_{2y,y} \end{bmatrix} = [Cd] \begin{bmatrix} \epsilon \\ \beta_{2x,y} \end{bmatrix} \quad (\text{II-11})$$

Sublamine 3:

Parameters  $J_{22}$ ,  $J_{26}$  and  $J_{66}$  in Equation (24) are defined by

$$J_{22} = D_{22}^0 + B_{22}^0 wd_{12} + B_{26}^0 wd_{22}$$

$$J_{26} = D_{26}^0 + B_{22}^0 wd_{13} + B_{26}^0 wd_{23}$$

$$J_{66} = D_{66}^0 + B_{26}^0 wd_{13} + B_{66}^0 wd_{23}$$

where

$$[wd]_{2 \times 3} = \begin{bmatrix} A_{22}^0 & A_{26}^0 \\ A_{26}^0 & A_{66}^0 \end{bmatrix}^{-1} \begin{bmatrix} A_{12}^0 & B_{22}^0 & B_{26}^0 \\ A_{16}^0 & B_{26}^0 & B_{66}^0 \end{bmatrix} \quad (\text{II-12})$$

Displacements  $U_3$  and  $V_3$  in sublamine 3 are related to the applied uniform strain  $\epsilon$  and the rotations  $\beta_{3x}$  and  $\beta_{3y}$  through the matrix  $wd$  by

$$\begin{bmatrix} v_3 \\ u_3 \end{bmatrix} = [wd] \begin{bmatrix} \epsilon \\ \beta_{3y,y} \\ \beta_{3x,y} \end{bmatrix} \quad (\text{II-13})$$

### Interlaminar Stresses:

Parameters  $\alpha_j$ ,  $\psi_j$  and  $\gamma_j$  in Equation (29) are defined as

$$\begin{bmatrix} \alpha_j \\ \psi_j \\ \gamma_j \end{bmatrix} = - \begin{bmatrix} (F_{31}s_j^2 - A_{45}^1) & F_{21}s_j^2 & F_{41}s_j^2 \\ (F_{33}s_j^2 - A_{55}^1) & F_{32}s_j^2 & F_{43}s_j^2 \\ (F_{32}s_j^2 & (F_{22}s_j^2 - A_{44}^0) & (F_{42}s_j^2 - A_{45}^0) \end{bmatrix}^{-1} \begin{bmatrix} (F_{11}s_j^2 - A_{44}^1) \\ (F_{31}s_j^2 - A_{45}^1) \\ F_{21}s_j^2 \end{bmatrix}$$

$$v_j = v_{11} + \alpha_j v_{13} + \psi_j v_{12} + \gamma_j v_{14}$$

$$u_j = u_{11} + \alpha_j u_{13} + \psi_j u_{12} + \gamma_j u_{14}$$

(II-14)

The parameters associated with the resultant force and moment distributions in sublaminae 1 and 0, Equation (30), are given by

$$\begin{bmatrix} N_{x1j} \\ N_{y1j} \\ N_{xy1j} \\ M_{y1j} \\ M_{xy1j} \end{bmatrix} = \begin{bmatrix} A_{12}^1 & A_{16}^1 & B_{12}^1 & B_{16}^1 \\ A_{22}^1 & A_{26}^1 & B_{22}^1 & B_{26}^1 \\ A_{26}^1 & A_{66}^1 & B_{26}^1 & B_{66}^1 \\ B_{22}^1 & B_{26}^1 & D_{22}^1 & D_{26}^1 \\ B_{26}^1 & B_{66}^1 & D_{26}^1 & D_{66}^1 \end{bmatrix} \begin{bmatrix} v_j \\ u_j \\ 1 \\ \alpha_j \end{bmatrix}$$

$$\begin{bmatrix} N_{xoj} \\ N_{y oj} \\ N_{xyoj} \\ M_{y oj} \\ M_{xyoj} \end{bmatrix} = \begin{bmatrix} A_{12}^o & A_{16}^o & (B_{12}^o - \frac{h_o}{2} A_{12}^o) & (B_{16}^o - \frac{h_o}{2} A_{16}^o) - \frac{h_1}{2} A_{12}^o - \frac{h_1}{2} A_{16}^o \\ A_{22}^o & A_{26}^o & (B_{22}^o - \frac{h_o}{2} A_{22}^o) & (B_{26}^o - \frac{h_o}{2} A_{26}^o) - \frac{h_1}{2} A_{22}^o - \frac{h_1}{2} A_{26}^o \\ A_{26}^o & A_{66}^o & (B_{26}^o - \frac{h_o}{2} A_{26}^o) & (B_{66}^o - \frac{h_o}{2} A_{66}^o) - \frac{h_1}{2} A_{26}^o - \frac{h_1}{2} A_{66}^o \\ B_{22}^o & B_{26}^o & (D_{22}^o - \frac{h_o}{2} B_{26}^o) & (D_{26}^o - \frac{h_o}{2} B_{26}^o) - \frac{h_1}{2} B_{26}^o - \frac{h_1}{2} B_{26}^o \\ B_{26}^o & B_{66}^o & (D_{26}^o - \frac{h_o}{2} B_{22}^o) & (D_{66}^o - \frac{h_o}{2} B_{66}^o) - \frac{h_1}{2} B_{26}^o - \frac{h_1}{2} B_{66}^o \end{bmatrix} \begin{bmatrix} v_j \\ u_j \\ \psi_j \\ \gamma_j \\ 1 \\ \alpha_j \end{bmatrix}$$

(II-15)

### Appendix III

#### Classical Lamination Theory (CLT) Solution

In the interior of the laminate CLT prevails that is, a membrane solution controls the behavior. The resultant forces  $N_y$  and  $N_{xy}$  are easily determined. In the following a ply-by-ply model is constructed.

The constitutive relationship for each ply for a membrane behavior is given by

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix}^k = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}^k \begin{bmatrix} \epsilon \\ \epsilon_y \\ \gamma_x \end{bmatrix}^k \quad (\text{III-1})$$

Superscript  $k$  denotes the  $k$ th ply. According to the present formulation

$$\epsilon_y = V(y),_y$$

$$\gamma_x = U(y),_y \quad (\text{III-2})$$

Since  $\epsilon_y$  and  $\gamma_x$  are functions of  $y$  only, therefore, from continuity of displacements at the interfaces between the  $k$ th and  $(k+1)$ th plies we have

$$\begin{aligned} u^k(y, \frac{h}{2}) &= u^{k+1}(y, \frac{-h^{k+1}}{2}) \\ v^k(y, \frac{h^k}{2}) &= v^{k+1}(y, \frac{-h^{k+1}}{2}) \end{aligned} \quad (\text{III-3})$$

Substitute for the displacements from Equation (1) into Equation (III-3) and differentiate w.r.t.  $y$  to get

$$\epsilon_y^k = \epsilon_y^{k+1} = \epsilon_y$$

$$\gamma_x^k = \gamma_x^{k+1} = \gamma_x$$

(III-4)

From symmetry, consider one quarter of the laminate. From equilibrium of forces get

$$\sum_{k=1}^N N_y^k = \sum_{k=1}^N N_{xy}^k = 0$$

(III-5)

Substitute from Equation (III-1) into (III-5) to get

$$s_{22} \epsilon_y + s_{26} \gamma_x + s_{12} \epsilon = 0$$

$$s_{26} \epsilon_y + s_{66} \gamma_x + s_{16} \epsilon = 0$$

where

$$s_{lm} = \sum_{k=1}^N A_{lm}^k \quad l = 1, 2 \quad m = 2 \text{ and } 6 \quad (III-6)$$

Solving for  $\epsilon_y$  and  $\gamma_x$  from Equation (III-6) to get

$$\epsilon_y = (s_{26}s_{16} - s_{66}s_{12})\epsilon/s$$

$$\gamma_x = (s_{26}s_{12} - s_{22}s_{16})\epsilon/s$$

$$s = s_{22}s_{66} - (s_{26})^2$$

(III-7)



Substituting from Equation (III-7) into Equation (III-1) to get

$$\begin{aligned} N_y^k &= (A_{12}^k \epsilon + A_{22}^k \epsilon_y + A_{26}^k \gamma_x) \\ N_{xy}^k &= (A_{16}^k \epsilon + A_{26}^k \epsilon_y + A_{66}^k \gamma_x) \end{aligned} \quad (\text{III-8})$$

By comparing the expressions for  $C_v$  and  $C_u$  in Equation (33) with Equation (III-7), we find that for sublaminates 1 and 0

$$C_v = \epsilon_y$$

and

$$C_u = \gamma_x \quad (\text{III-9})$$

Therefore the first term in Equation (30) for  $N_x$ ,  $N_y$  and  $N_{xy}$  in sublaminates 1 and 0 is the CLT prediction.

From equilibrium considerations the sign of the resultant axial force  $N_y^k$  in each ply determines the sign of the interlaminar peel stress. This is illustrated for a  $[0/45/90/-45]_s$  with a delamination at the 0/45 interface. With reference to Figure 26, this layup corresponds to a  $[45/90/-45/0]_s$  laminate rotated at an angle  $\theta = -45^\circ$ .

For the properties shown in Table I and using Equations (III-7) and (III-8) the resultant forces  $N_y$  and  $N_{xy}$  in each ply can be determined. These are shown in Table XI.

Consider the free body diagram of the top ply shown in Figure 29. From equilibrium of forces in the vertical direction the peel stress distribution should reverse its sign such that its resultant is zero. Furthermore, the peel stress at the crack tip should be compressive in order to balance the moment of the compressive resultant force  $N_y$ .

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Table I. Material Properties and Geometry of the ED Specimen

$$\begin{aligned}
 E_{11} &= 134 \text{ GPa} \\
 E_{22} &= 10.2 \text{ GPa} \\
 G_{12} &= 5.52 \text{ GPa} \\
 \nu_{12} &= 0.3 \\
 G_{31} &= G_{23} = 3.1 \text{ GPa} \\
 \text{Ply thickness (h)} &= 0.14 \times 10^{-3} \text{ m} \\
 \text{Semi-Width (b)} &= 140h \\
 \text{Crack length (a)} &= 6h
 \end{aligned}$$

Table II. Comparison of Characteristic Equation Coefficients

Laminate	$E_8 \times 10^{-9}$	$E_6 \times 10^{-9}$	$E_4 \times 10^{-9}$	$E_2 \times 10^{-9}$	$E_0 \times 10^{-9}$
$[15/60/-75/-30]_s$	0.143	$-6.940 \times 10^3$	$9.089 \times 10^7$	$-3.132 \times 10^{11}$	$3.193 \times 10^{14}$
$[-35/55/10/-80]_s$	2.282	$-6.049 \times 10^4$	$4.614 \times 10^8$	$-1.155 \times 10^{12}$	$5.676 \times 10^{14}$
$[40/-50/85/-5]_s$	0.095	$-6.389 \times 10^3$	$1.071 \times 10^8$	$-4.096 \times 10^{11}$	$3.193 \times 10^{14}$

Table III. Comparison of the Characteristic Roots for a  $[35/80/-55/-10]_s$  Layup

Model	$s_{x_1}$	$s_{y_1}$	$s_{x_2}$	$s_{y_2}$
Coupled	5547	4286	2714	2438
Uncoupled	5538	3737	2427	2108
Membrane			2802	2162

Table IV. COMPARISON OF ENERGY RELEASE COMPONENTS IN  
ED SPECIMENS (a = 6h, b = 140h)

STACKING SEQUENCE	$G_I/G_T$		$G_{II}/G_T$		$G_{III}/G_T$	
	PRESENT	FEM	PRESENT	FEM	PRESENT	FEM
[35/-35/0/90] <sub>s</sub>	86.5	90.0	13.3	10.0	0.2	0.0
[35/0/-35/90] <sub>s</sub>	59.5	58.0	40.5	42.0	0.0	0.0
[0/35/-35/90] <sub>s</sub>	26.3	22.0	73.7	78.0	0.0	0.0
[30/-60/75/-15] <sub>s</sub>	9.6	16.9	4.9	0.4	85.5	81.7
[35/80/-55/-10] <sub>s</sub>	13.5	7.5	1.9	2.7	84.6	89.8
[-35/55/10/-80] <sub>s</sub>	18.5	20.6	14.8	11.7	66.7	67.7

Table V.  $[0/90/\pm 45]_s$  Laminate with Circular Hole- Edge Delamination at  
+ 45/-45 Interface ( $a = 6h$ ,  $b = 140h$ ,  $\varepsilon = 10^{-6}$ )

ANGULAR POSITION	STACKING SEQUENCE	$G_I/G_T$		$G_{II}/G_T$		$G_{III}/G_T$		$10^6 G_T$	
		PRESENT	FEM	PRESENT	FEM	PRESENT	FEM	PRESENT	FEM
50°	$[40/-50/85/-5]_s$	57.65	53.65	1.03	2.96	41.31	43.39	2.83	2.93
55°	$[35/-55/80/-10]_s$	30.59	24.11	2.33	3.45	67.08	72.44	4.94	5.11
60°	$[30/-60/75/-15]_s$	18.97	10.98	4.19	5.21	76.85	83.80	6.29	6.51
65°	$[25/-65/70/-20]_s$	10.47	3.40	6.81	7.86	82.72	88.74	6.73	6.95

Table VI.  $[0/90/\pm 45]_S$  Laminate with Circular Hole- Edge Delamination at  
90/45 Interface ( $a = 6h$ ,  $b = 140h$ ,  $\epsilon = 10^{-6}$ )

ANGULAR POSITION	STACKING SEQUENCE	$G_I/G_T$		$G_{II}/G_T$		$G_{III}/G_T$		$10^6 G_T$	
		PRESENT	FEM	PRESENT	FEM	PRESENT	FEM	PRESENT	FEM
-70°	$[-20/70/25/-65]_S$	10.33	1.80	1.01	-2.20	88.66	100.40	6.11	6.37
-65°	$[-25/65/20/-70]_S$	12.59	5.80	0.17	-3.18	87.24	97.38	6.11	6.36
-60°	$[-30/60/15/-75]_S$	15.11	11.87	3.73	0.86	81.16	87.26	5.42	5.64
-55°	$[-35/55/10/-80]_S$	18.56	20.64	14.79	11.66	66.65	67.71	4.10	4.27
-50°	$[-40/50/5/-85]_S$	24.34	33.18	42.16	36.48	33.50	30.34	2.60	2.70
-45°	$[-45/45/0/90]_S$	27.87	35.07	72.13	65.40	0.00	-0.46	1.88	1.95

Table VII. [0/90/  $\pm$  45] Laminate with Circular Hole- Edge Delamination at  
0/90 Interface ( $a = 6h$ ,  $b = 140h$ ,  $\varepsilon = 10^{-6}$ )

ANGULAR POSITION	STACKING SEQUENCE	$G_I/G_T$		$G_{II}G_T^{\infty}$		$G_{III}/G_T$		$10^6 G_T$	
		PRESENT	FEM	PRESENT	FEM	PRESENT	FEM	PRESENT	FEM
-85°	$[-5/85/40/-50]_s$	3.40	0.22	0.08	0.34	96.52	99.44	1.26	1.40
-80°	$[-10/80/35/-55]_s$	4.08	0.40	0.40	0.54	95.52	99.06	3.41	3.74
-75°	$[-15/75/30/-60]_s$	5.06	0.86	0.98	1.07	93.96	98.07	4.88	5.33
-70°	$[-20/70/25/-65]_s$	6.29	1.59	1.85	1.97	91.86	96.43	5.51	6.03
-65°	$[-25/65/20/-70]_s$	7.73	2.49	3.04	3.36	89.24	94.15	5.53	6.03
-60°	$[-30/60/15/-75]_s$	9.25	3.37	4.55	5.35	86.21	91.29	5.10	5.49
-55°	$[-35/55/10/-80]_s$	10.62	4.07	6.34	7.95	83.04	87.98	4.30	4.56
-50°	$[-40/50/5/-85]_s$	11.58	4.52	8.26	10.92	80.16	84.56	3.27	3.36
-45°	$[-45/45/0/90]_s$	11.89	4.64	9.87	13.48	78.23	81.88	2.14	2.10



Table VIII. [45/90/-45/0]<sub>s</sub> Laminate with Circular Hole- Edge Delamination at  
-45/0 Interface (a = 6h, b = 140h,  $\epsilon = 10^{-6}$ )

ANGULAR POSITION	STACKING SEQUENCE	$G_I/G_T$		$G_{II}/G_T$		$G_{III}/G_T$		$10^6 G_T$	
		PRESENT	FEM	PRESENT	FEM	PRESENT	FEM	PRESENT	FEM
-85°	[ 40/85/-50/-5] <sub>s</sub>	2.57	2.68	0.00	-1.28	97.43	98.61	1.28	1.25
-80°	[ 35/80/-55/-10] <sub>s</sub>	0.07	2.28	0.00	-1.27	99.93	98.99	3.43	3.47
-75°	[ 30/75/-60/-15] <sub>s</sub>	0.57	1.81	0.00	-1.09	99.43	99.27	4.91	5.00
-70°	[ 25/70/-65/-20] <sub>s</sub>	1.14	1.25	0.41	-0.44	98.44	99.19	5.55	5.68
-65°	[ 20/65/-70/-25] <sub>s</sub>	1.71	0.77	1.47	0.98	96.81	98.25	5.56	5.72
-60°	[ 15/60/-75/-30] <sub>s</sub>	2.19	0.51	3.49	3.46	94.32	96.03	5.10	5.27
-55°	[ 10/55/-80/-35] <sub>s</sub>	2.61	0.56	6.85	7.24	90.54	92.20	4.30	4.46
-50°	[ 5/50/-85/-40] <sub>s</sub>	3.65	1.35	11.88	12.45	84.48	86.20	3.30	3.43
-45°	[ 0/45/90/-45] <sub>s</sub>	8.18	4.68	18.11	19.12	73.71	76.20	2.31	2.40
-40°	[ -5/40/85/-50] <sub>s</sub>	24.36	17.55	21.87	24.38	53.77	58.07	1.56	1.62
-35°	[ -10/35/80/-55] <sub>s</sub>	59.94	47.44	14.92	21.58	25.14	30.98	1.20	1.25
-30°	[ -15/30/75/-60] <sub>s</sub>	95.57	79.57	0.00	10.90	4.43	9.63	1.22	1.27

Table IX. [45/90/-45/0] Laminate with Circular Hole- Edge Delamination at  
90/-45 Interface ( $a = 6h$ ,  $b = 140h$ ,  $\epsilon = 10^{-6}$ ).

ANGULAR POSITION	STACKING SEQUENCE	$G_I/G_T$		$G_{II}/G_T$		$G_{III}/G_T$		$10^6 G_T$	
		PRESENT	FEM	PRESENT	FEM	PRESENT	FEM	PRESENT	O'BRIEN
85°	[50/-85/-40/5] <sub>s</sub>	7.46	3.12	53.34	52.36	38.2	44.52	1.88	2.03
90°	[45/90/-45/0] <sub>s</sub>	11.16	7.18	24.00	23.37	64.84	69.45	2.32	2.43
-85°	[40/85/-50/-5] <sub>s</sub>	12.94	8.42	7.14	7.84	79.92	83.75	3.20	3.37
-80°	[35/80/-55/-10] <sub>s</sub>	13.52	7.54	1.91	2.76	84.56	89.71	4.30	4.66
-75°	[30/75/-60/-15] <sub>s</sub>	13.74	6.02	0.98	1.67	85.28	92.30	5.28	5.83
-70°	[25/70/-65/-20] <sub>s</sub>	13.74	4.52	0.46	1.63	85.80	93.85	5.86	6.49
-65°	[20/65/-70/-25] <sub>s</sub>	12.34	3.38	0.00	1.52	87.66	95.20	5.88	6.40
-60°	[15/60/-75/-30] <sub>s</sub>	8.76	2.43	0.00	0.70	91.24	96.87	5.31	5.83
-55°	[10/55/-80/-35] <sub>s</sub>	4.64	2.13	0.00	-0.94	95.36	98.82	4.26	4.66
-50°	[5/50/-85/-40] <sub>s</sub>	5.63	2.73	0.00	-0.62	94.37	97.89	3.08	3.37
-45°	[0/45/90/-45] <sub>s</sub>	2.42	5.31	21.43	11.18	76.15	83.85	2.23	2.43
-40°	[-5/40/85/-50] <sub>s</sub>	16.78	10.03	51.48	38.02	31.74	51.95	1.89	2.03
-35°	[-10/35/80/-55] <sub>s</sub>	20.92	19.22	68.30	58.69	10.69	22.09	1.90	2.01
-45°	[-15/30/75/-60] <sub>s</sub>	29.51	32.61	67.58	61.72	2.92	5.68	2.00	2.10



Table XI. Resultant Forces for a  $[0/45/90/-45]_s$  Laminate

Ply Angle	$N_y$ (N/m)	$N_{xy}$ (N/m)
0	-0.008	0
45	2.671	3.031
90	-5.335	0
-45	2.671	-3.031

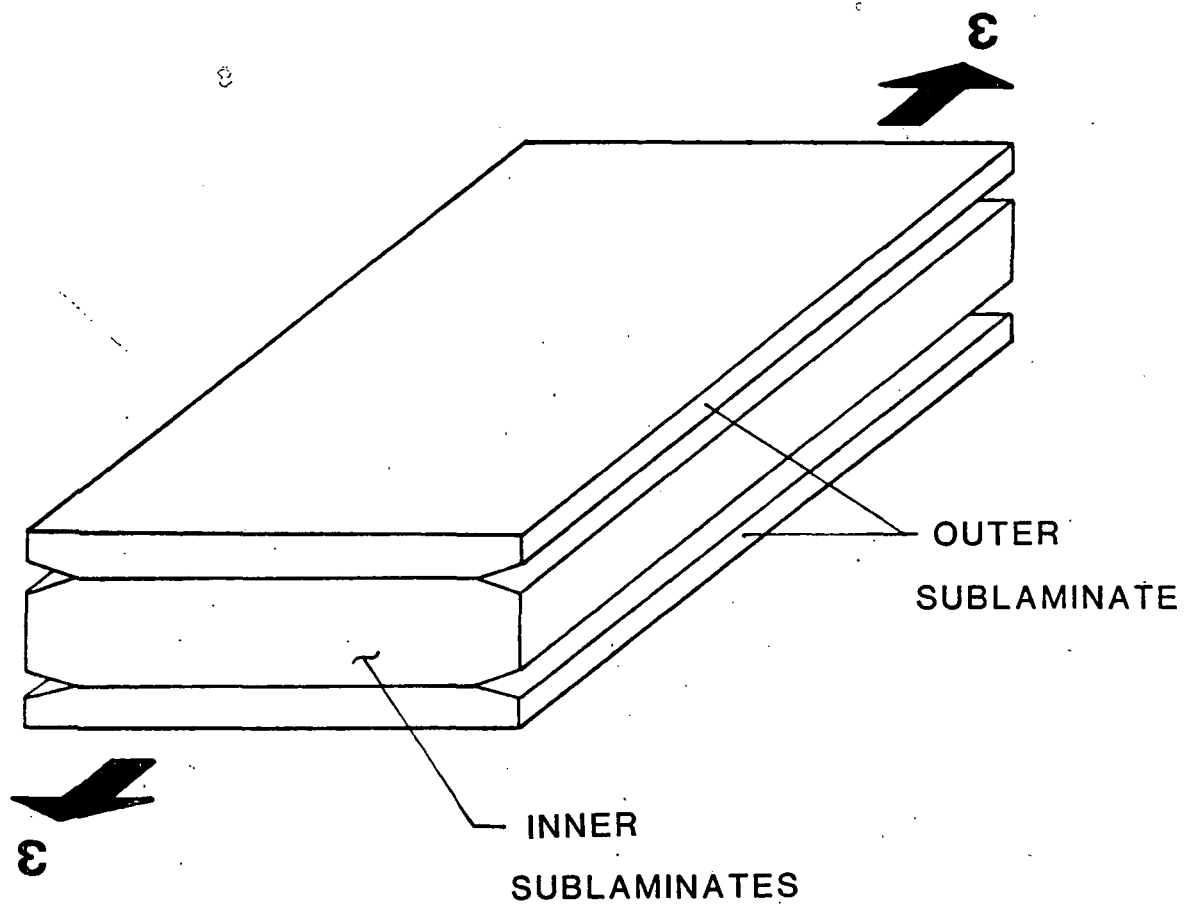


FIGURE 1. THE ED SPECIMEN

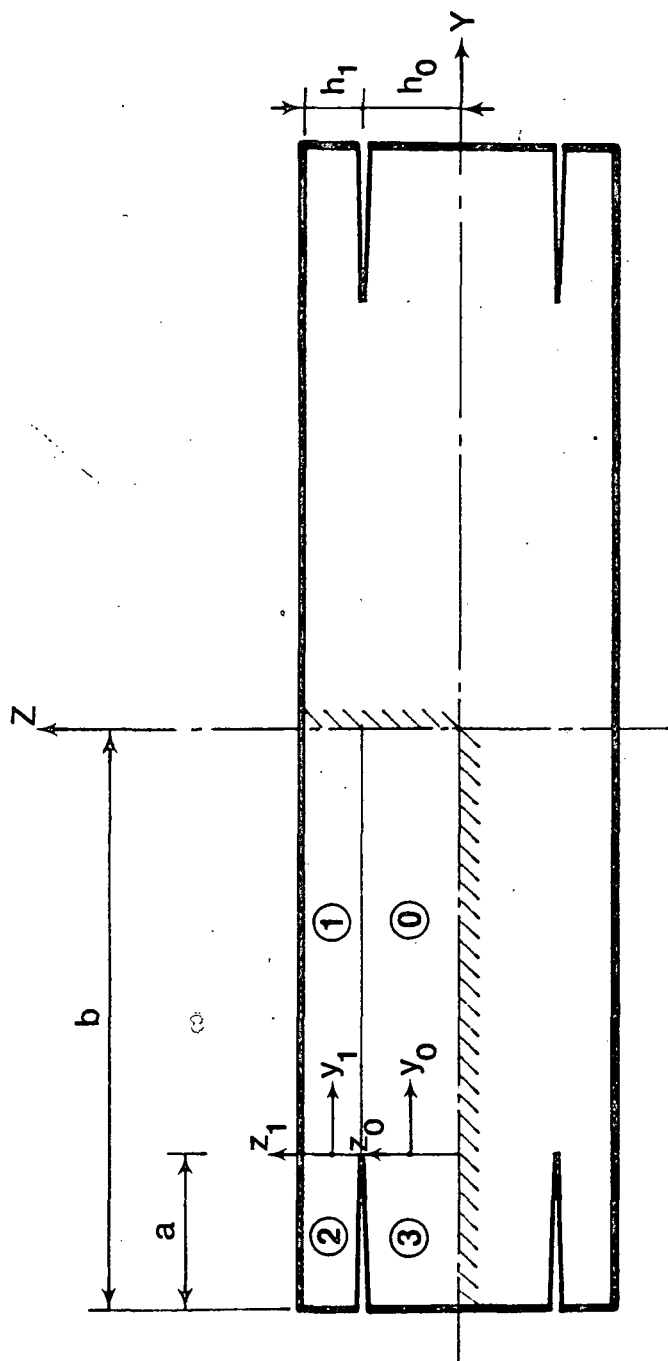


FIGURE 2. SUBLAMINATE DESCRIPTION AND COORDINATE SYSTEMS

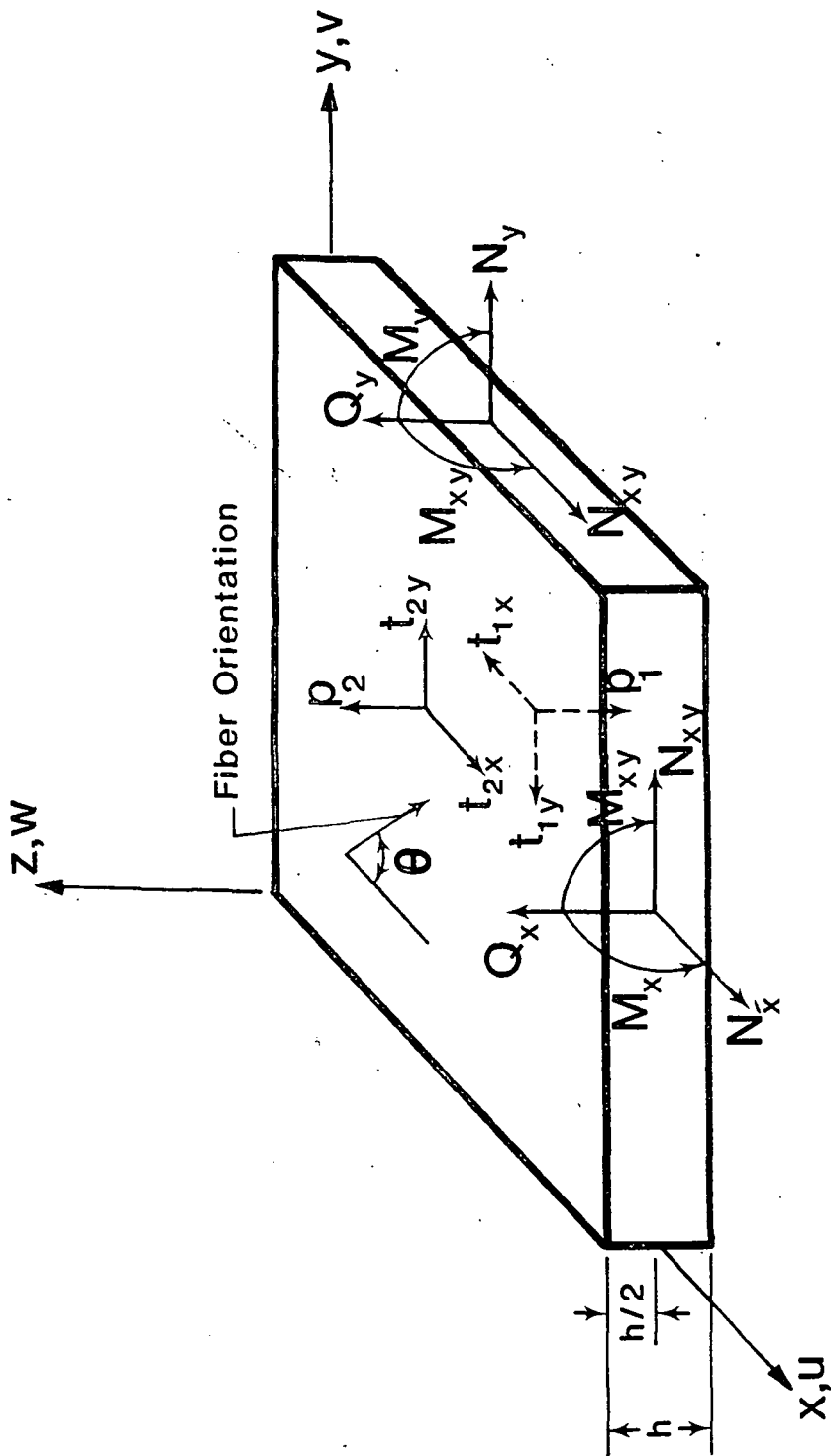


FIGURE 3. NOTATIONS AND SIGN CONVENTION FOR A GENERIC SUBLAMINATE

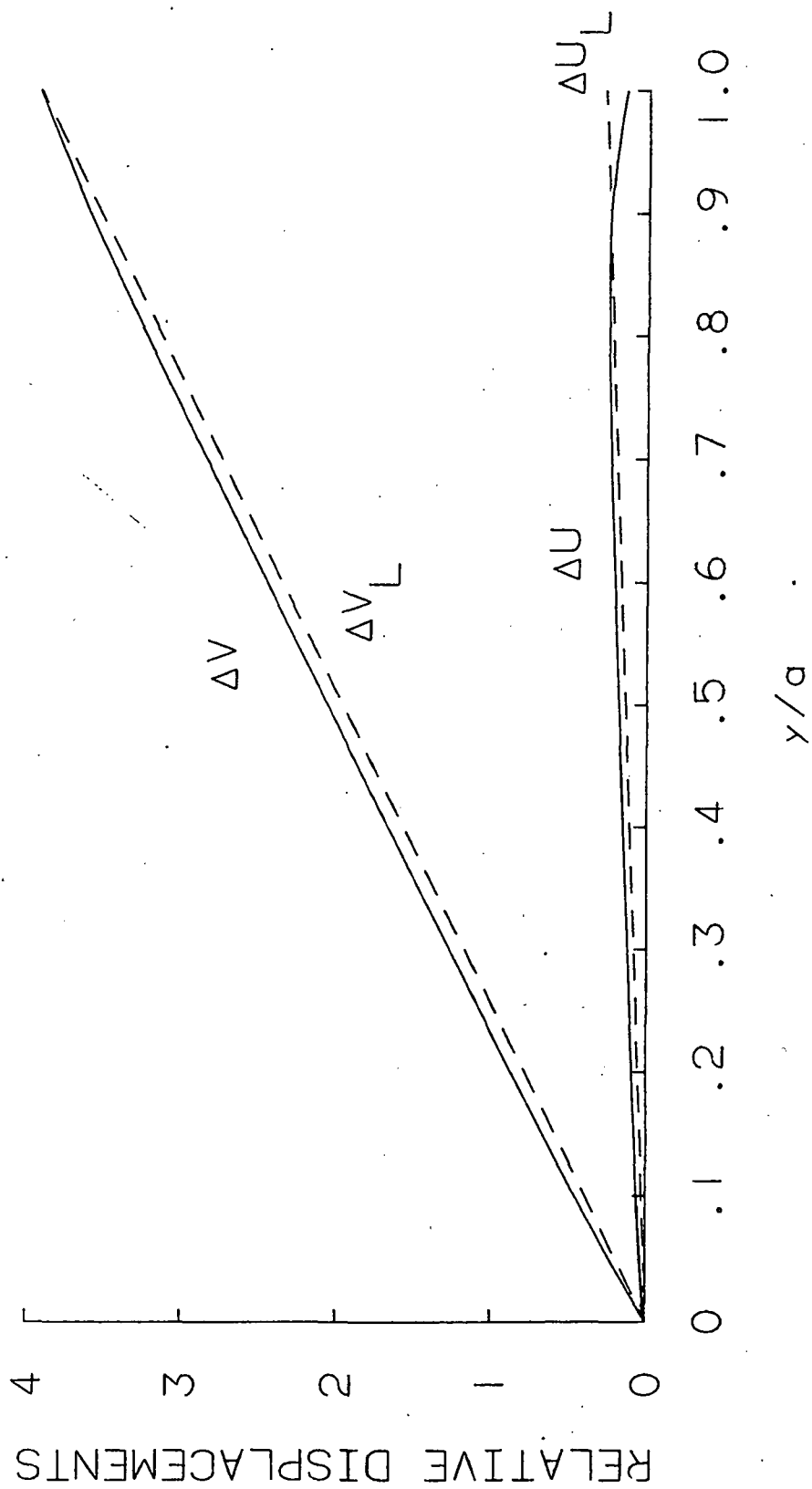


FIGURE. 4 COMPARISON OF RELATIVE DISPLACEMENTS AT THE CRACK SURFACE FOR A  $[0/35/-35/90]_s$  LAMINATE



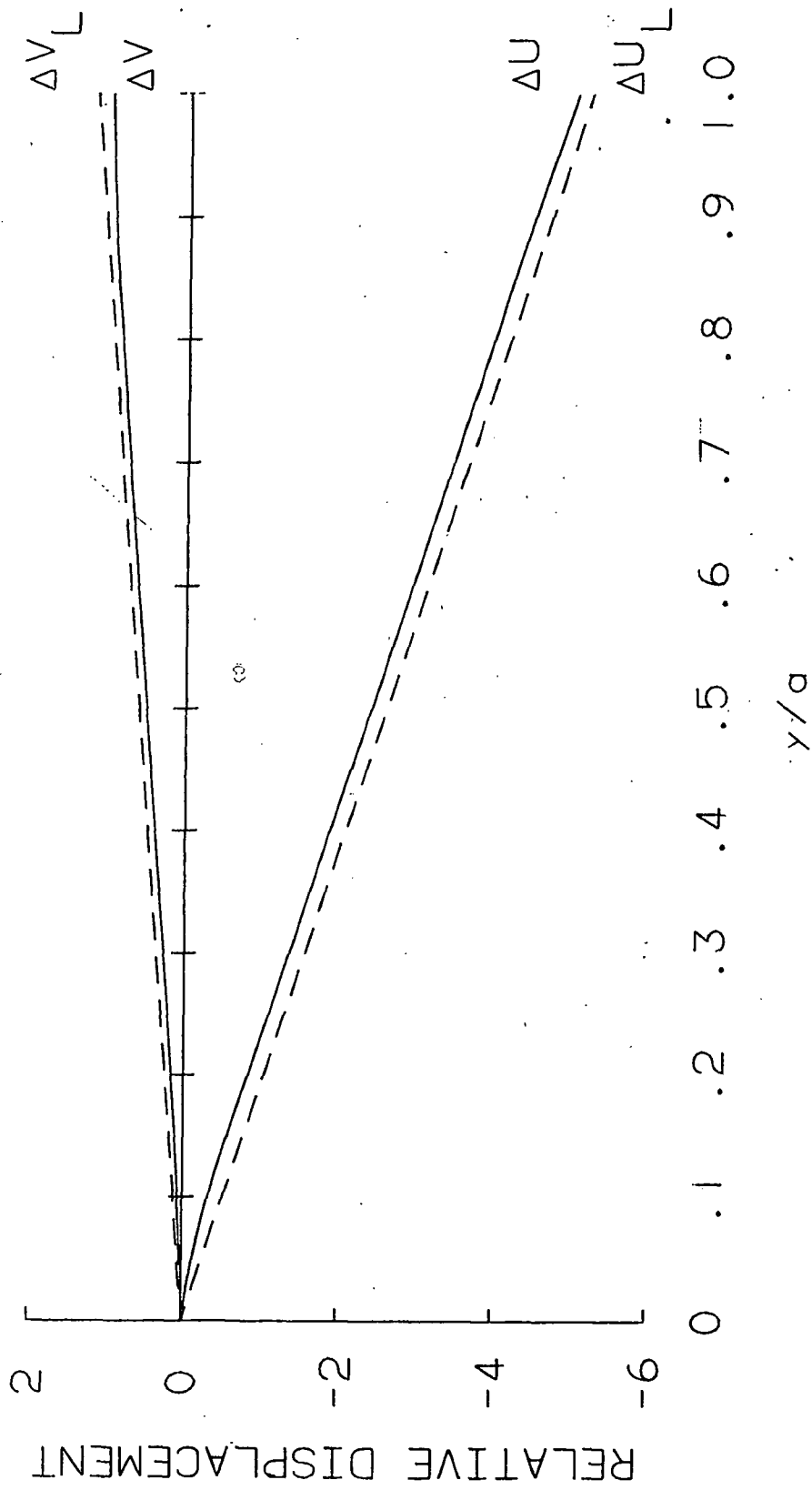


FIGURE 5. COMPARISON OF RELATIVE DISPLACEMENTS AT THE CRACK SURFACE FOR A  $[-35/55/10/-80]_s$  LAMINATE

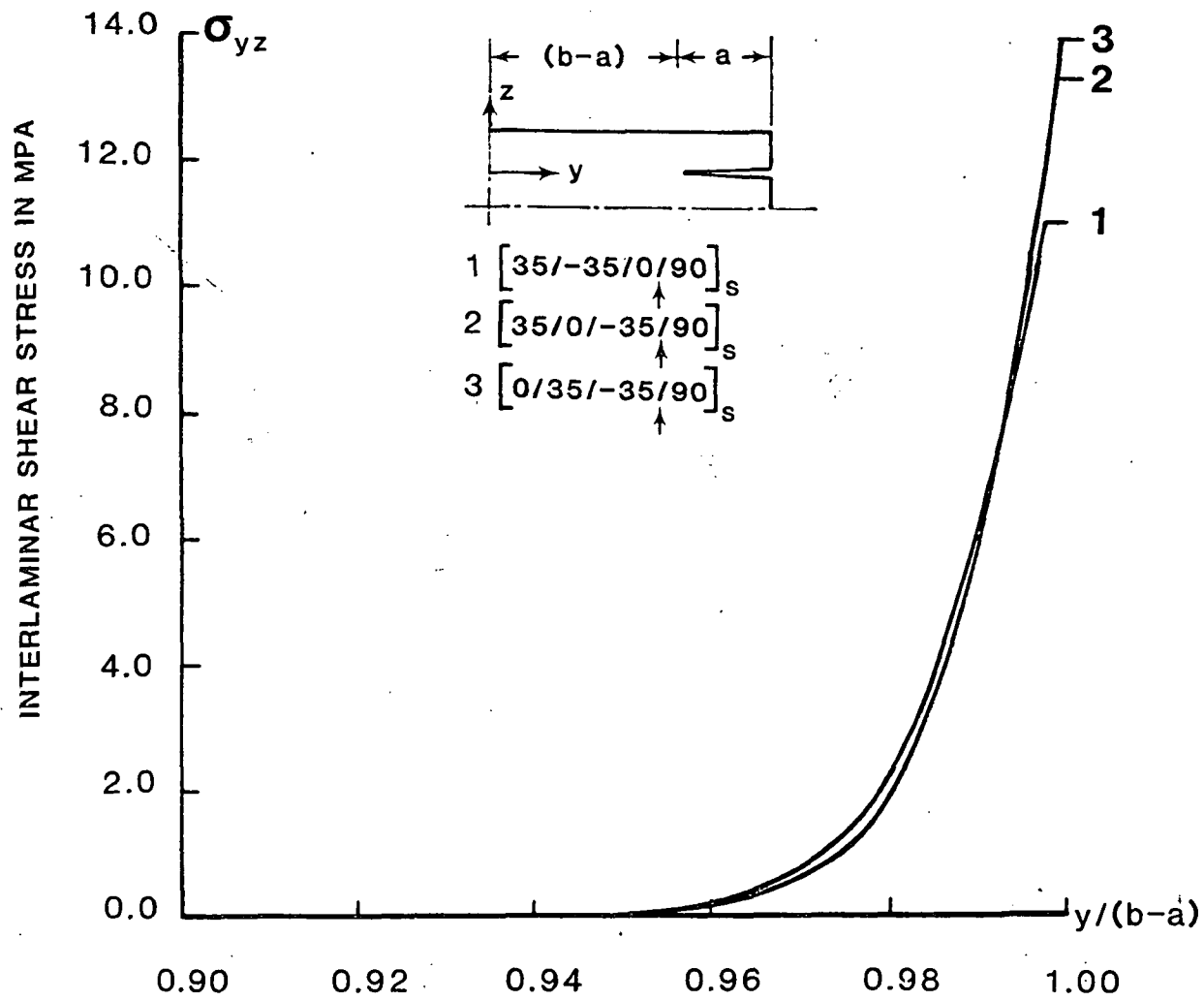


FIGURE 6. COMPARISON OF INTERLAMINAR SHEAR STRESS

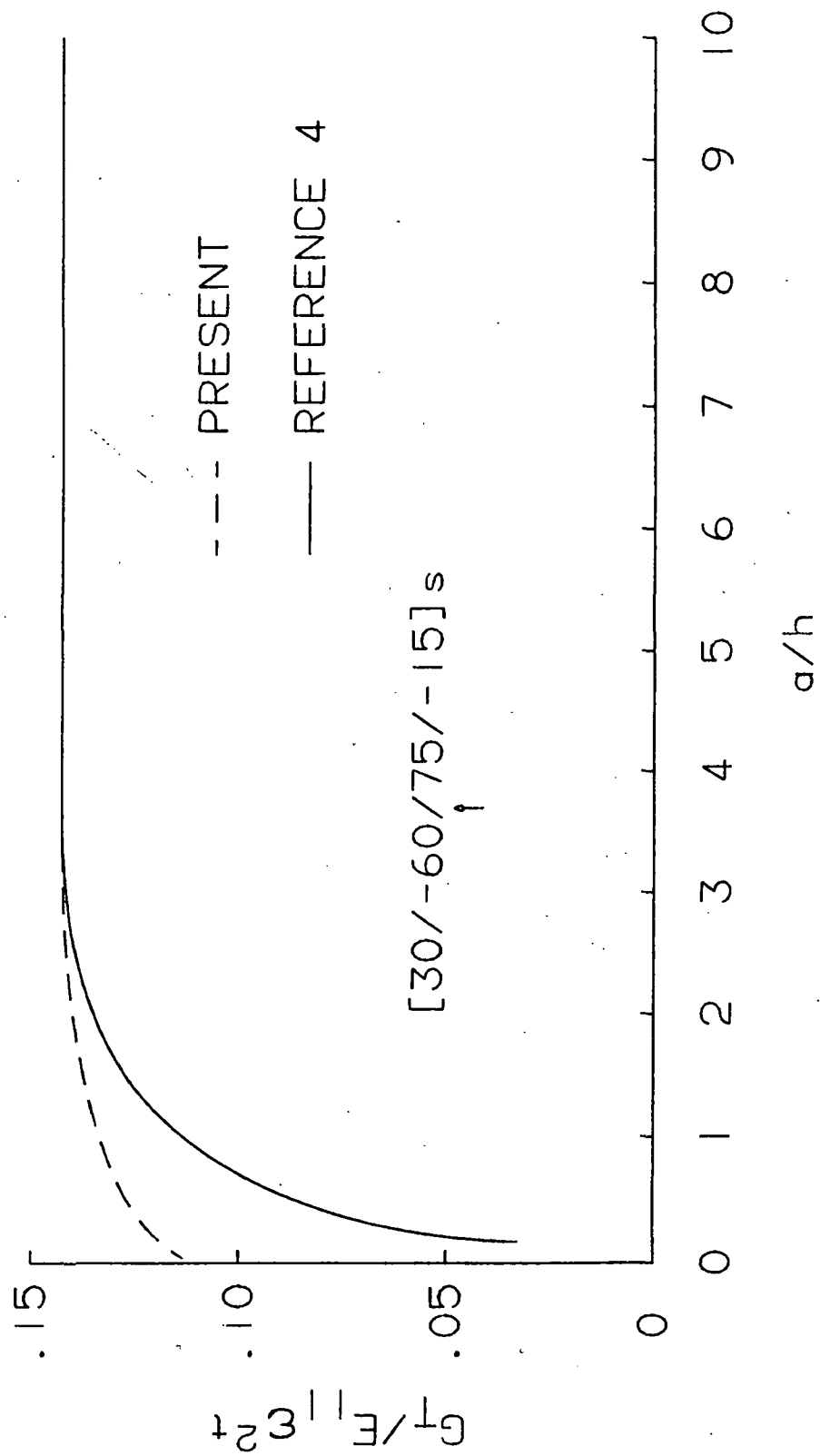


FIGURE 7. NORMALIZED STRAIN-ENERGY-RELEASE RATE AS A FUNCTION OF DELAMINATION SIZE DIVIDED BY PLY THICKNESS

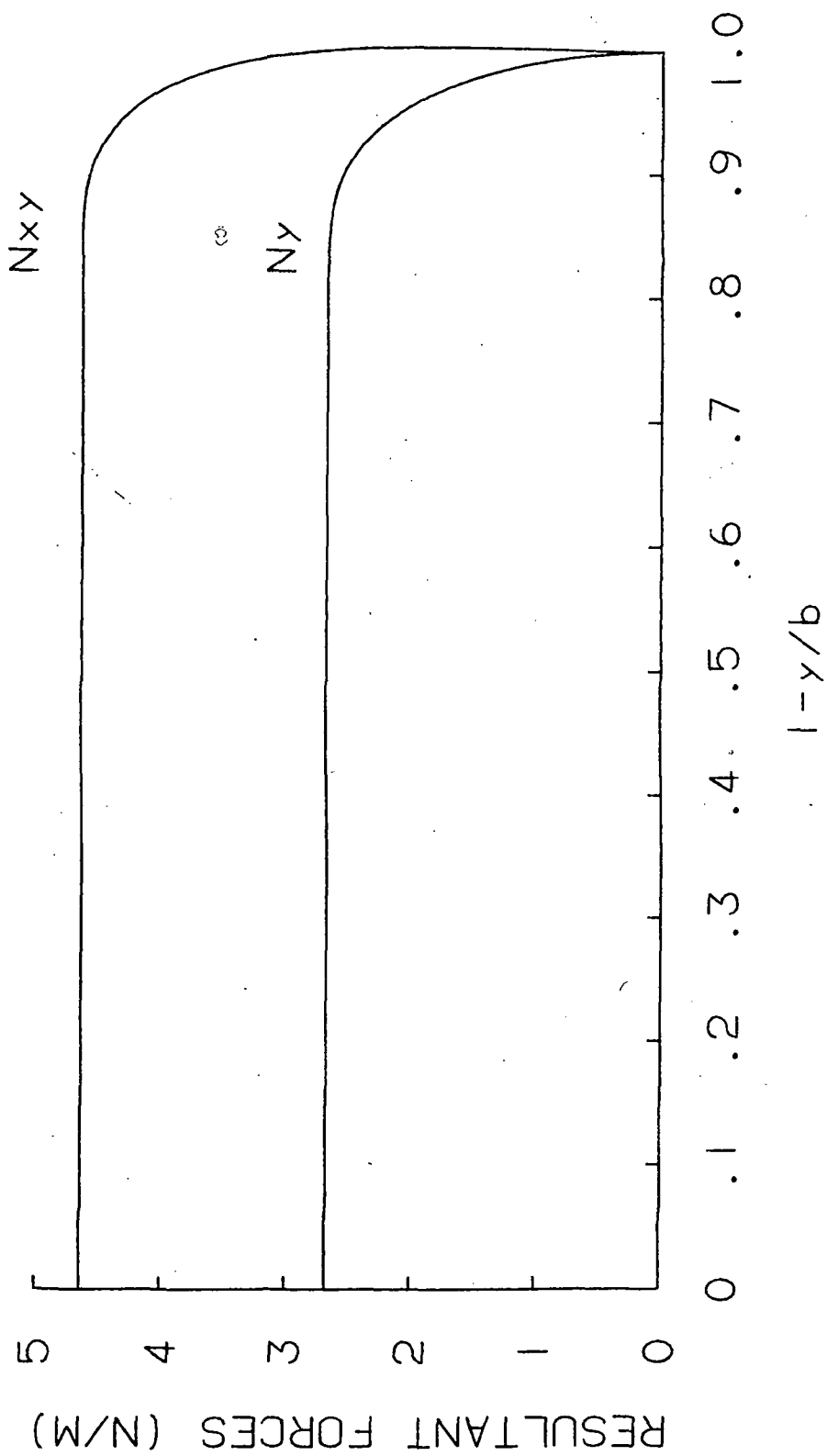


FIGURE 8. RESULTANT FORCE DISTRIBUTION  
IN A (30/-60/75/-15) LAMINATE

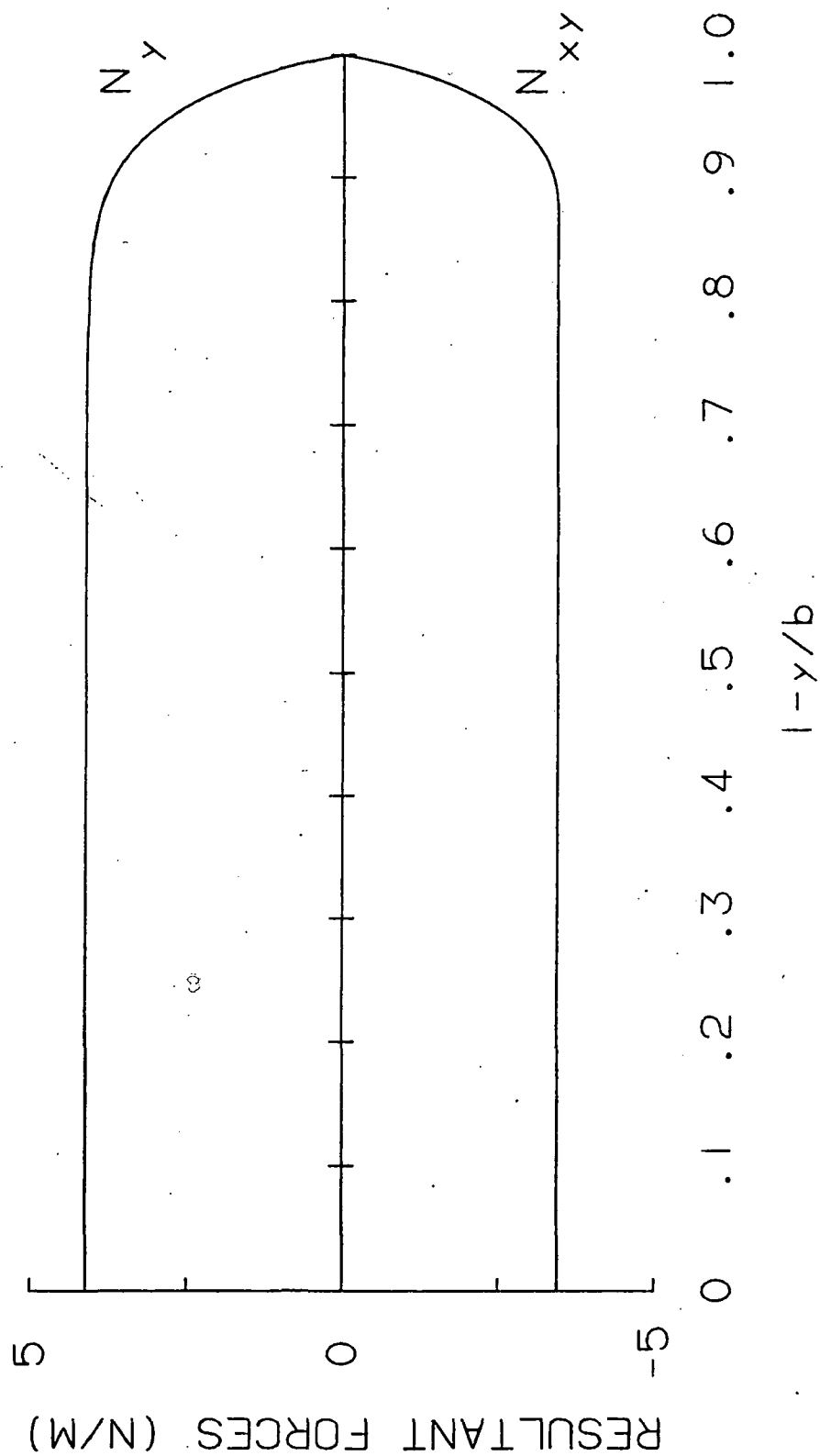


FIGURE 9. RESULTANT FORCE DISTRIBUTION  
IN A  $(-35/55/10/80)_S$  LAMINATE

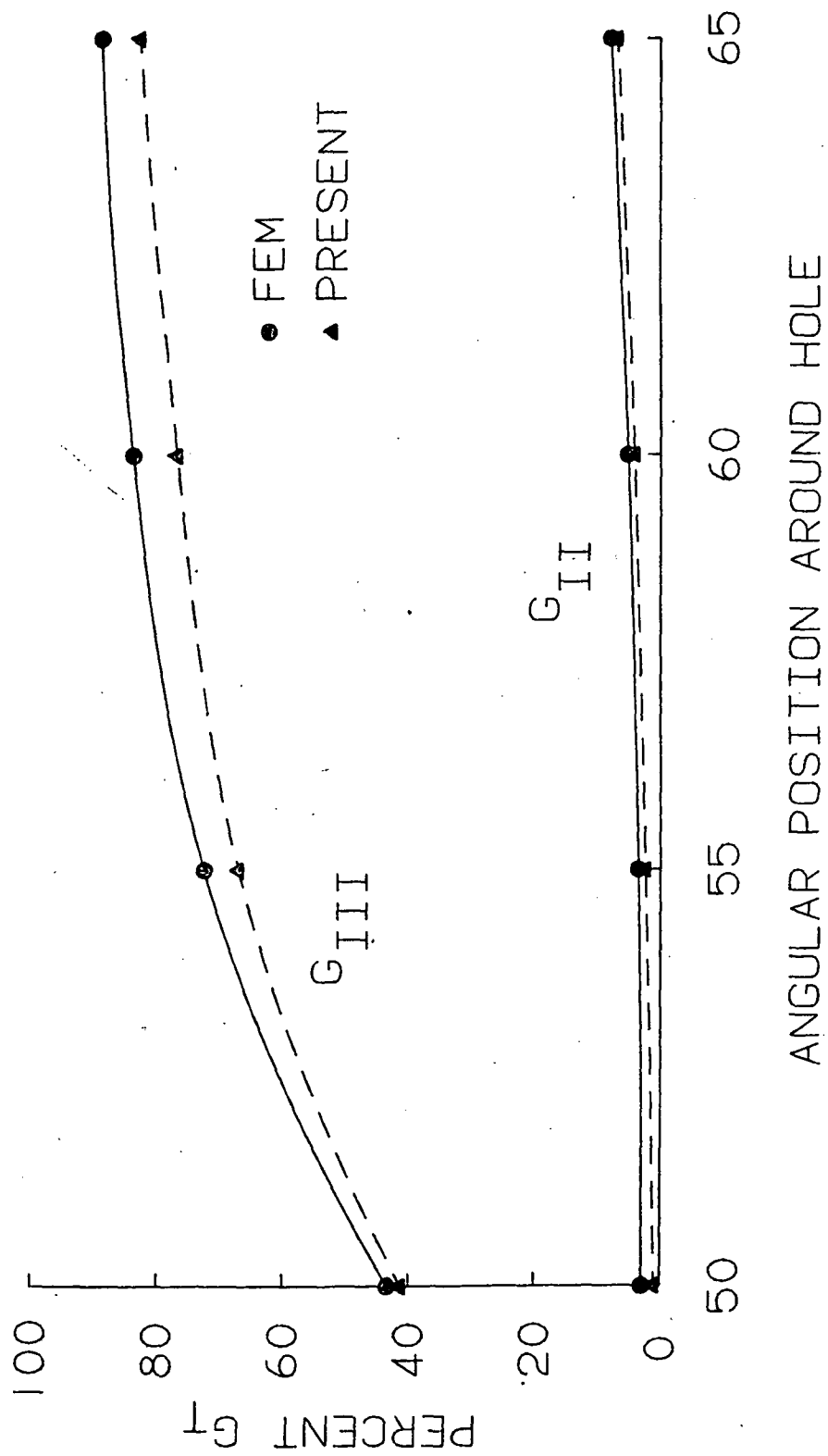


FIGURE 10.  $[0/90/\pm 45]_s$  LAMINATE WITH CIRCULAR HOLE  
EDGE DELAMINATION AT  $+45/-45$  INTERFACE

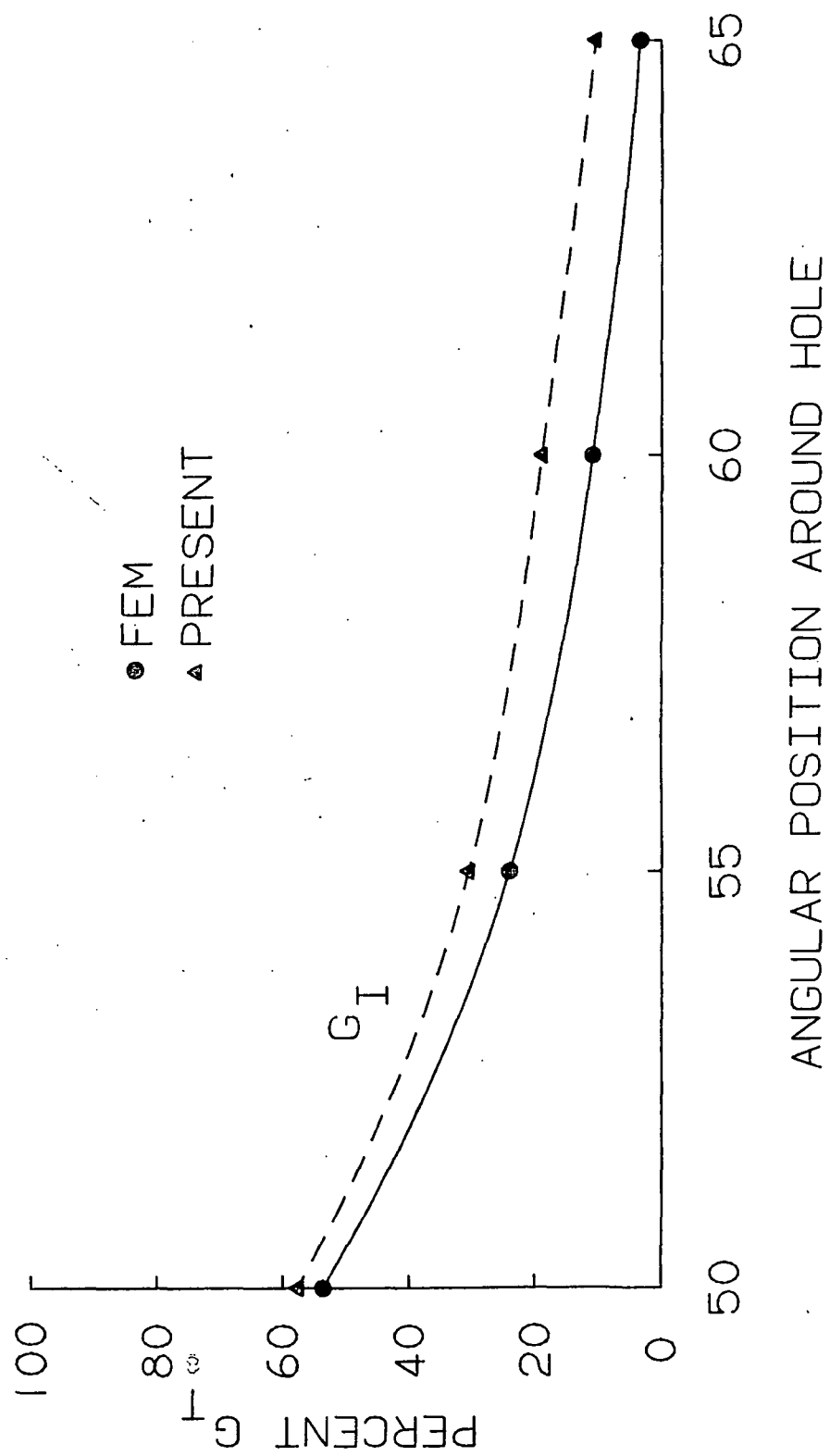


FIGURE 11.  $[0/90/\pm 45]_s$  LAMINATE WITH CIRCULAR HOLE  
EDGE DELAMINATION AT  $+45/-45$  INTERFACE

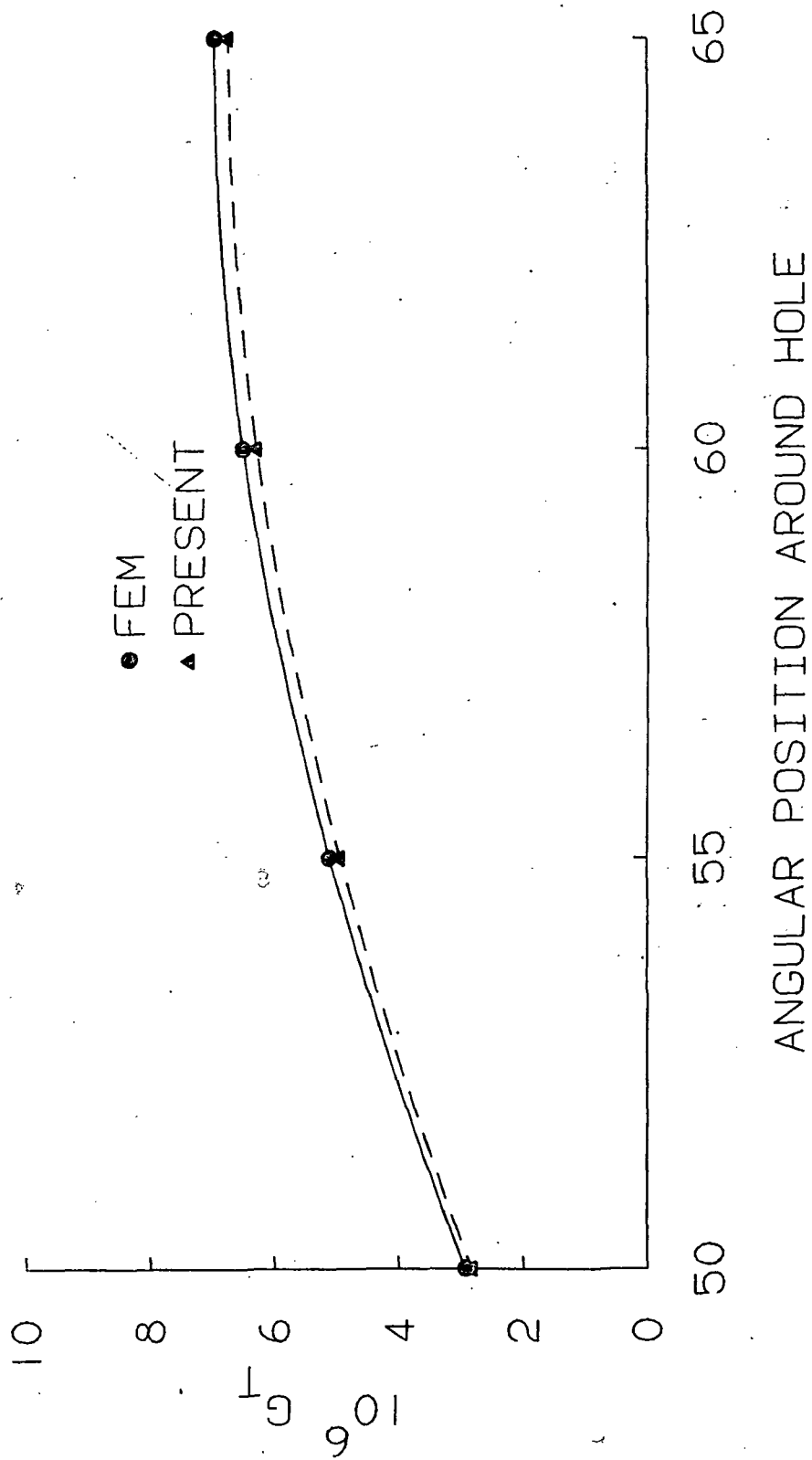


FIGURE 12.  $[0/90/\pm 45]_s$  LAMINATE WITH CIRCULAR HOLE  
EDGE DELAMINATION AT  $+45/-45$  INTERFACE



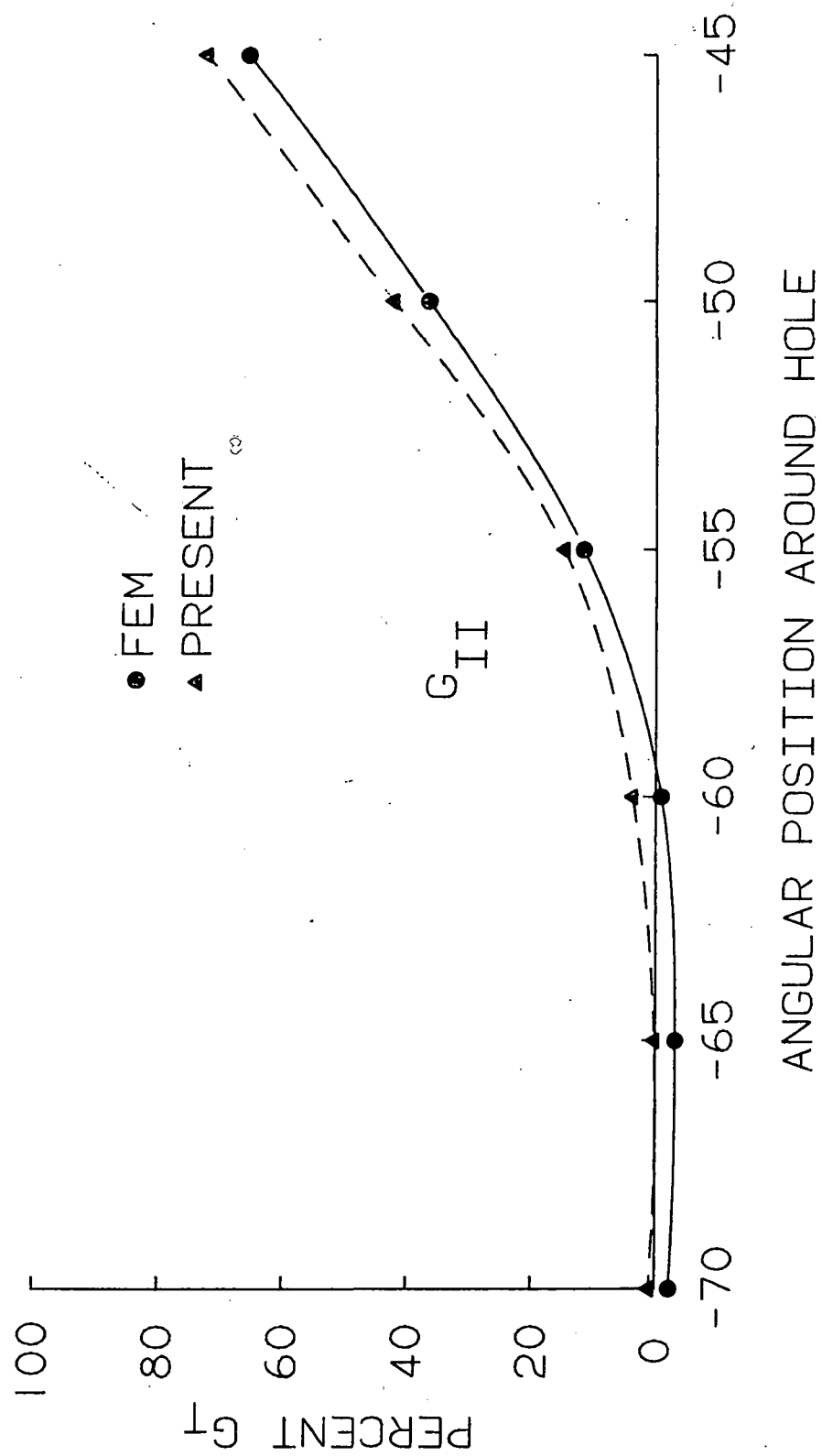


FIGURE 13.  $[0/90/\pm 45]_s$  LAMINATE WITH CIRCULAR HOLE  
EDGE DELAMINATION AT  $90^\circ/\pm 45^\circ$  INTERFACE

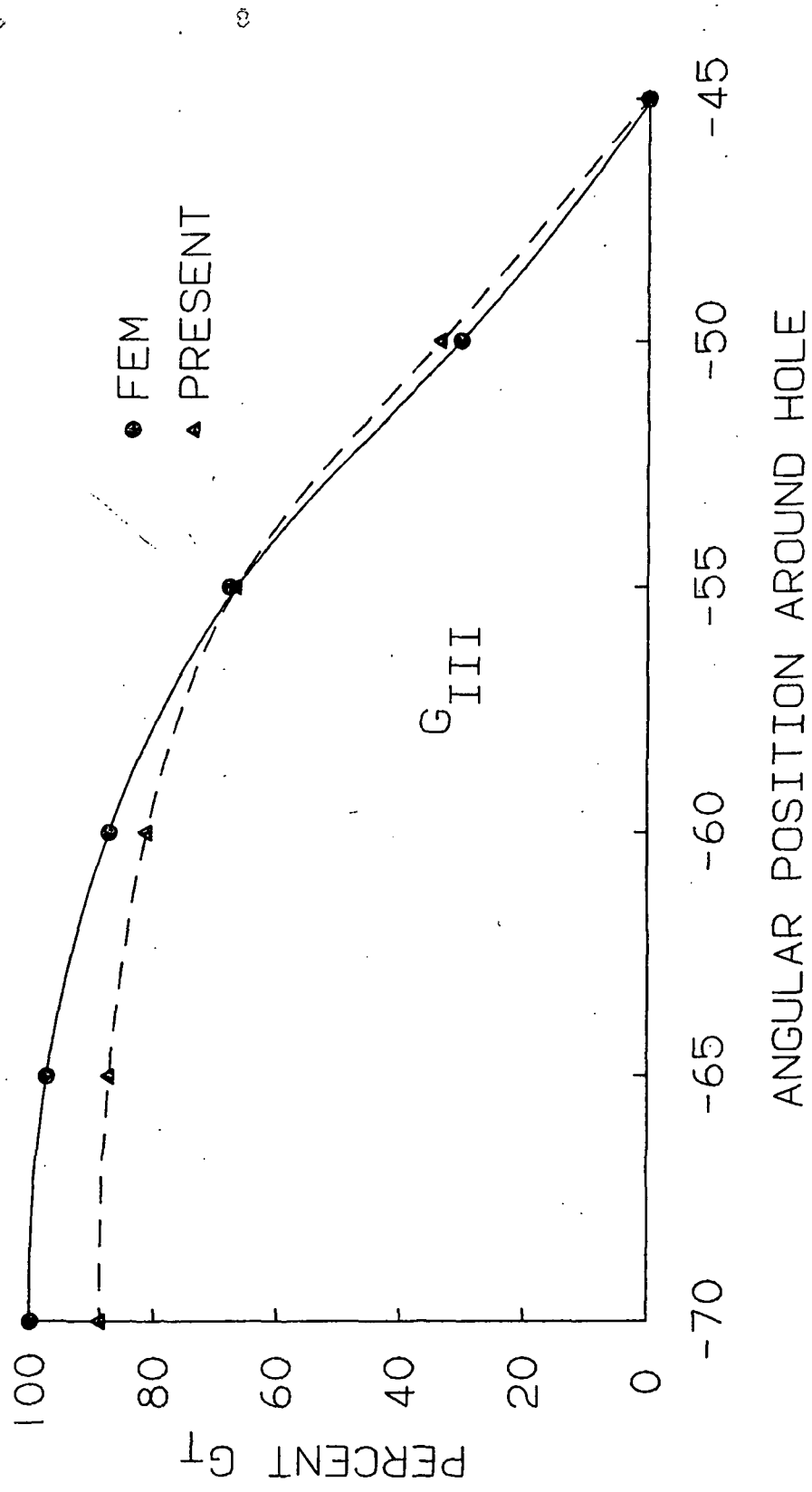


FIGURE 14.  $[0/90/+45]_s$  LAMINATE WITH CIRCULAR HOLE  
EDGE DELAMINATION AT  $90/+45$  INTERFACE

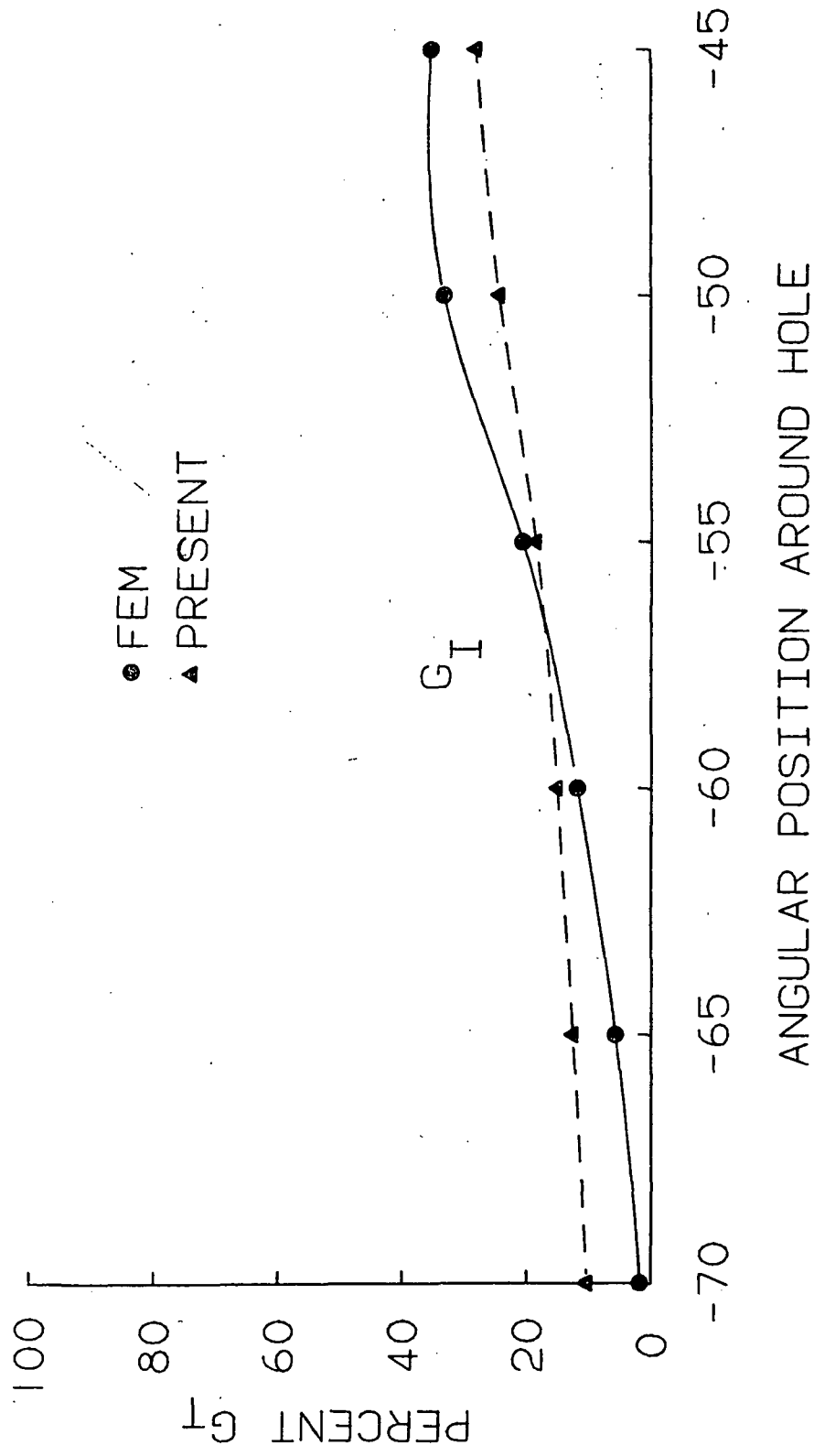


FIGURE 15.  $[0/90/\pm 45]_s$  LAMINATE WITH CIRCULAR HOLE  
EDGE DELAMINATION AT  $90^\circ/+45^\circ$  INTERFACE

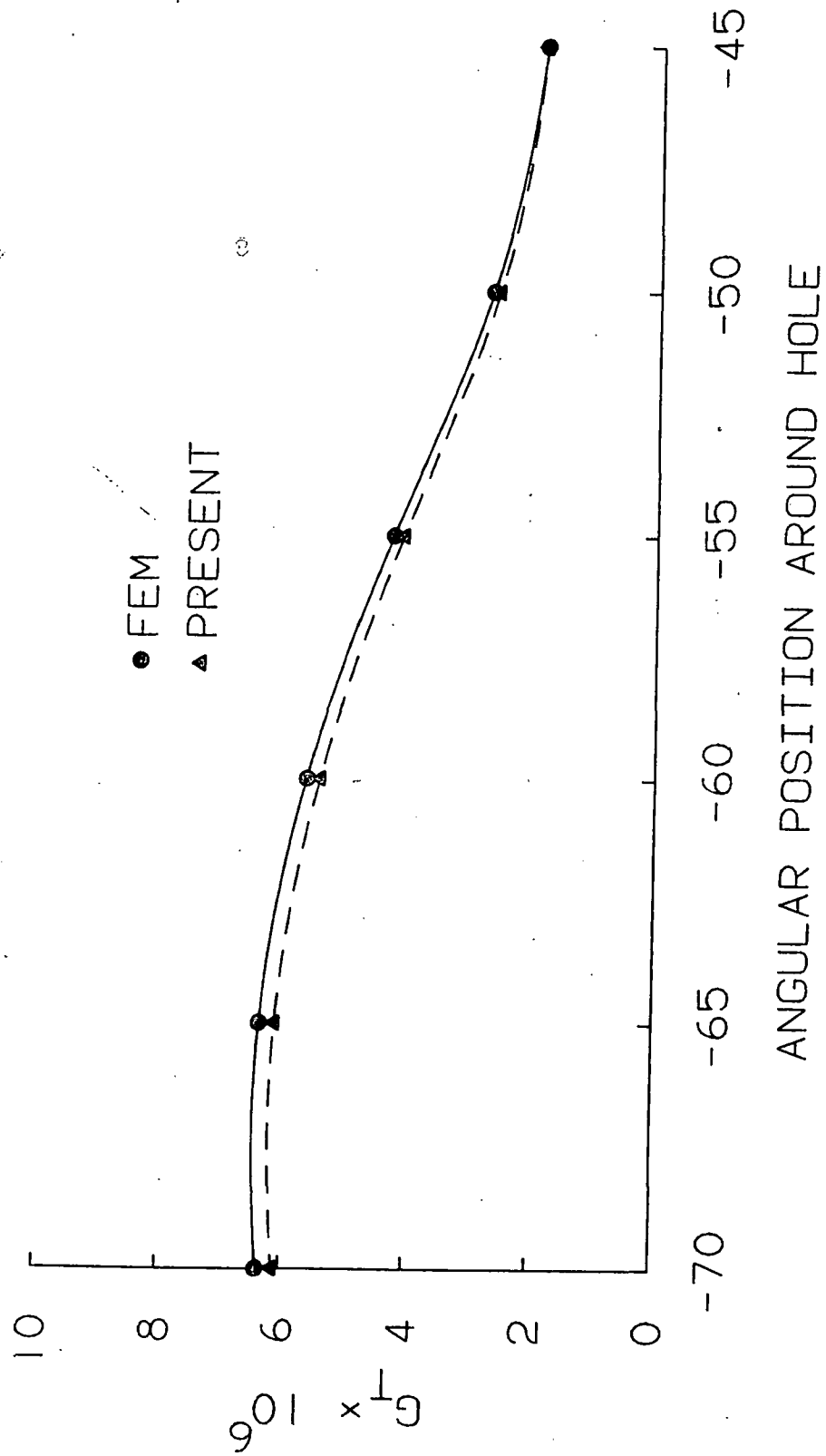


FIGURE 16.  $[0/90/\pm 45]_s$  LAMINATE WITH CIRCULAR HOLE  
EDGE DELAMINATION AT  $90/+45$  INTERFACE

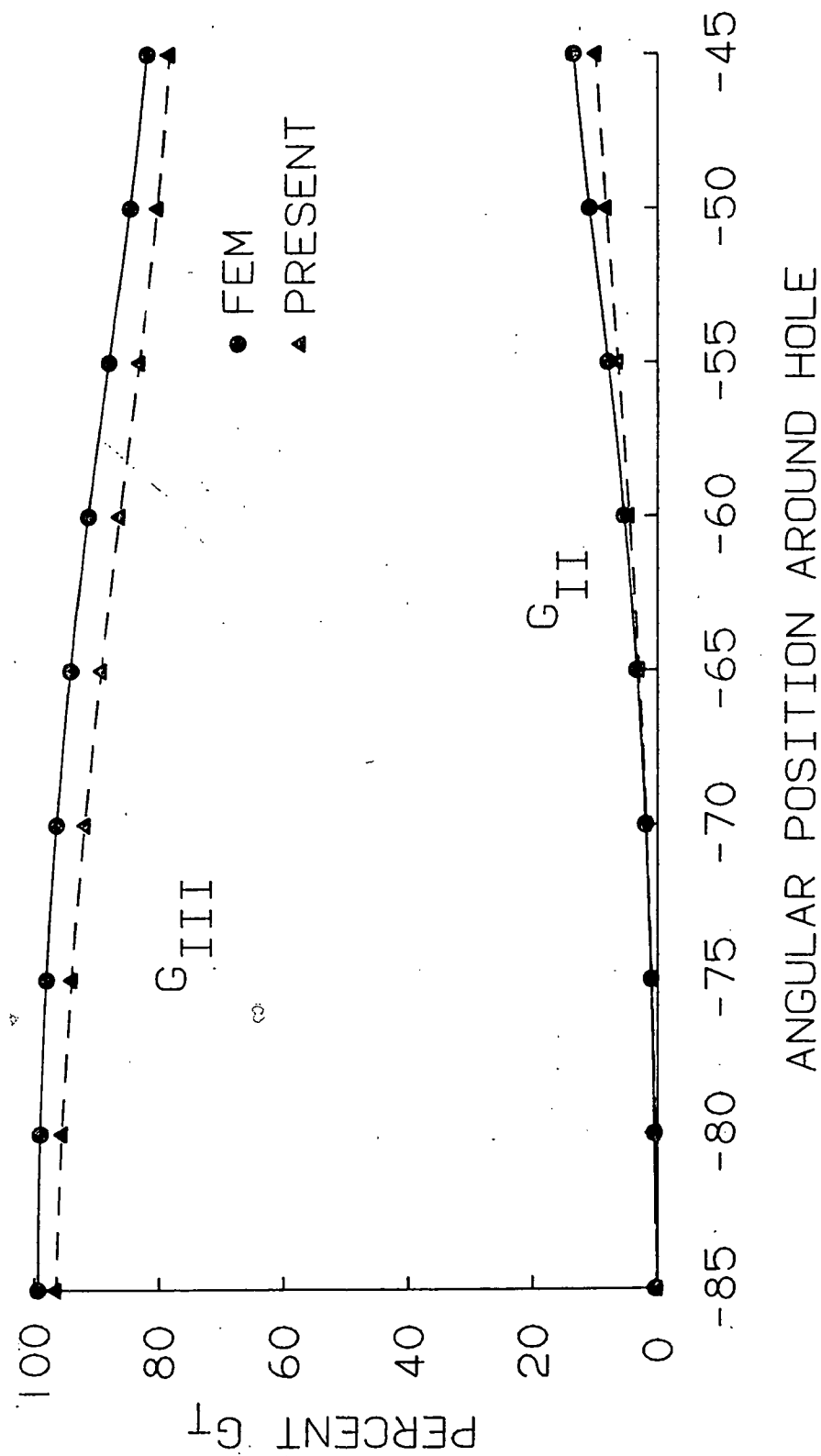


FIGURE 17.  $[0/90/\pm 45]_s$  LAMINATE WITH CIRCULAR HOLE  
EDGE DELAMINATION AT  $0/90$  INTERFACE

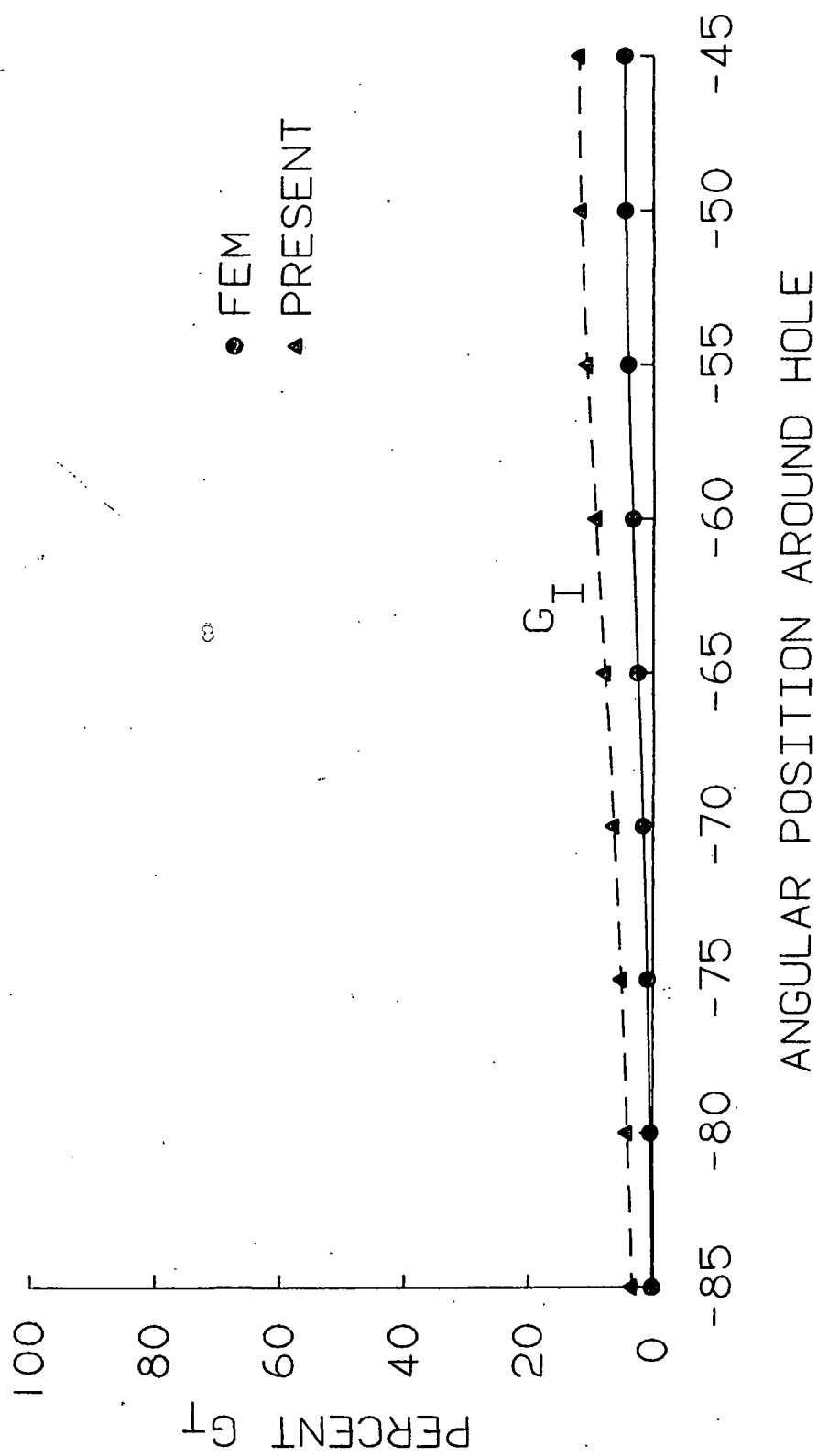


FIGURE 18.  $[0/90/\pm 45]_s$  LAMINATE WITH CIRCULAR HOLE  
EDGE DELAMINATION AT 0/90 INTERFACE

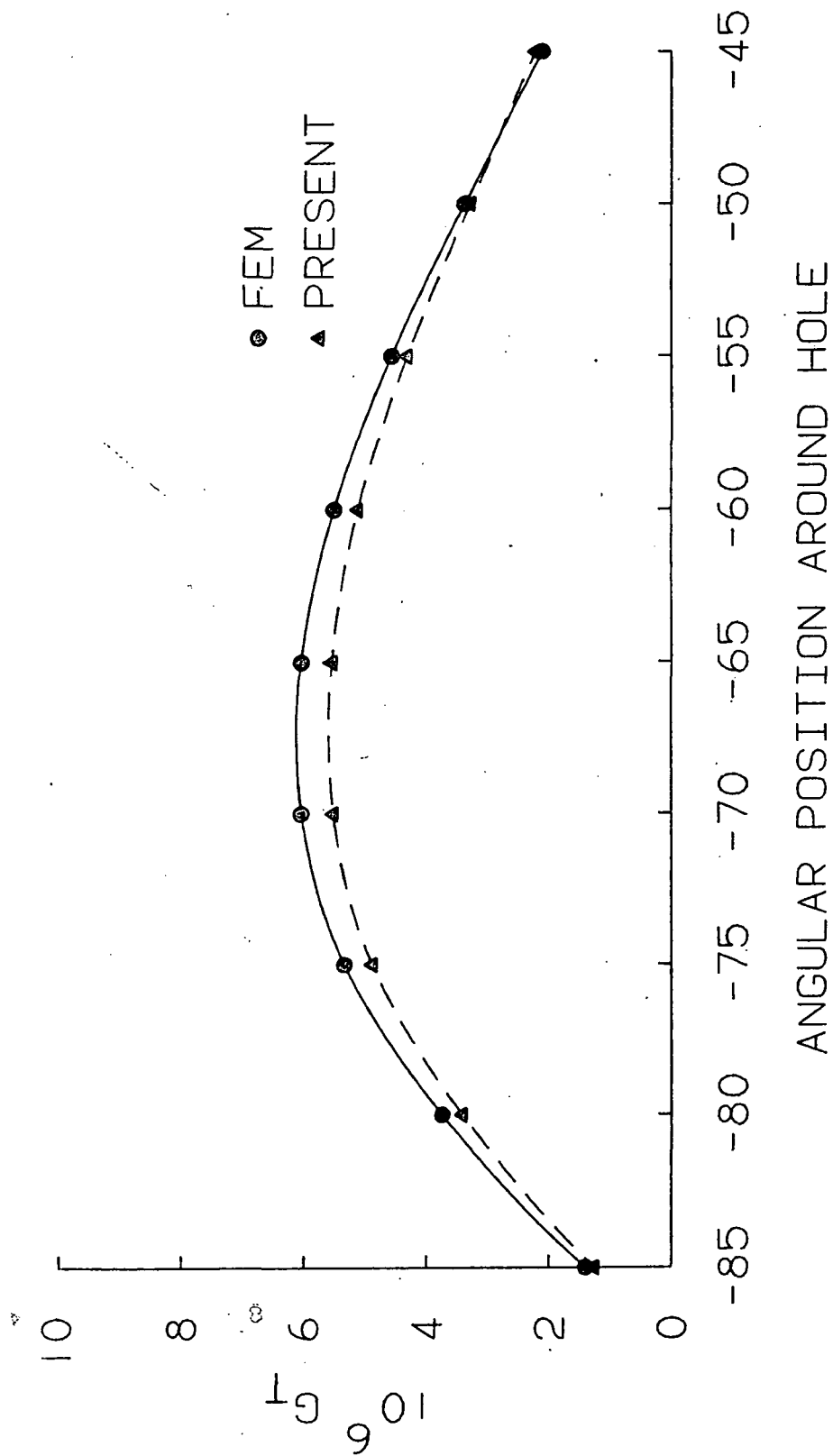


FIGURE 19.  $[0/90/\pm 45]_s$  LAMINATE WITH CIRCULAR HOLE  
EDGE DELAMINATION AT 0/90 INTERFACE

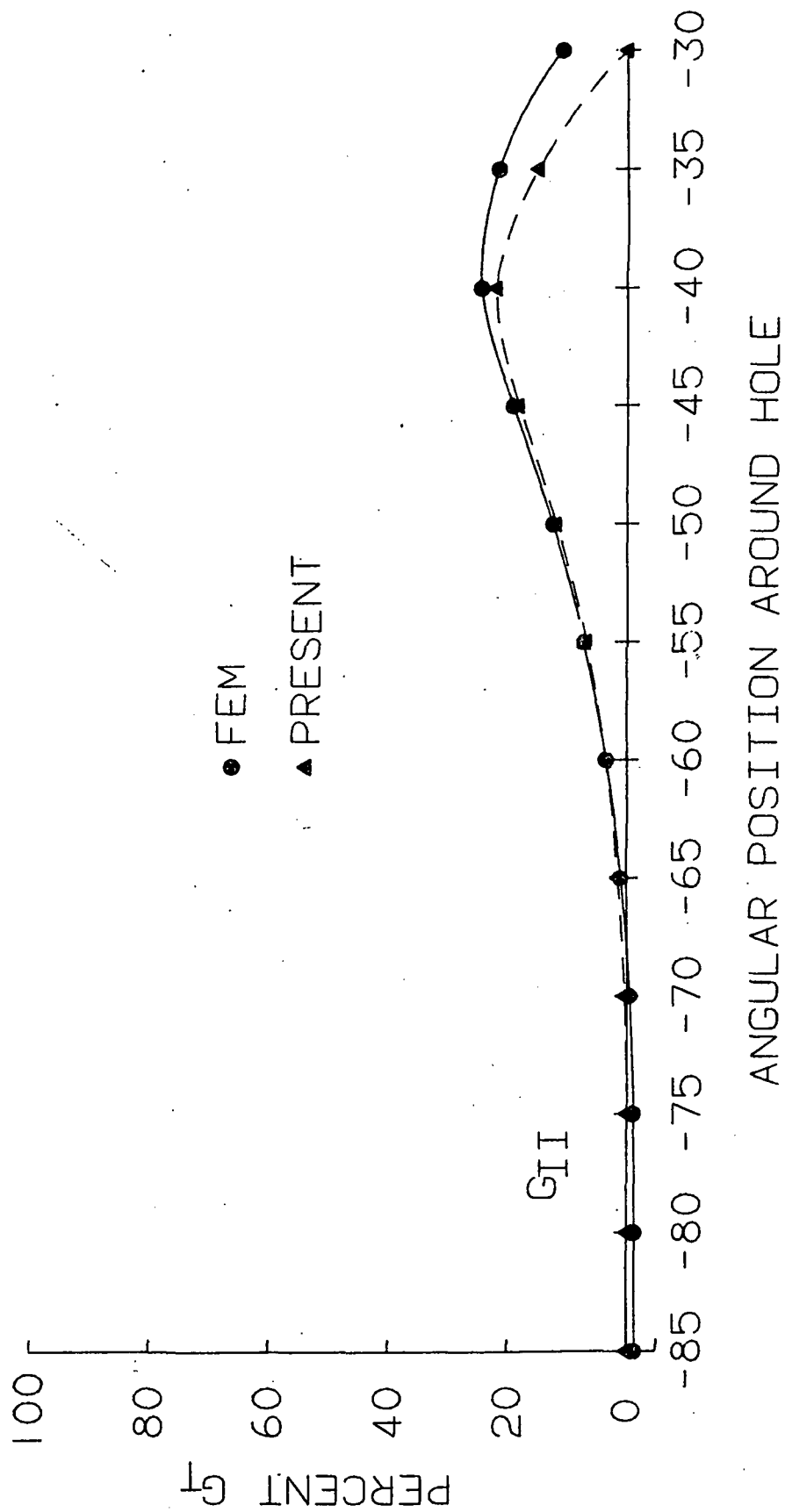


FIGURE 20. COMPARISON OF ENERGY RELEASE RATE FOR A  
[45/90/-45/0]<sub>s</sub> DELAMINATED AT -45/0 INTERFACE



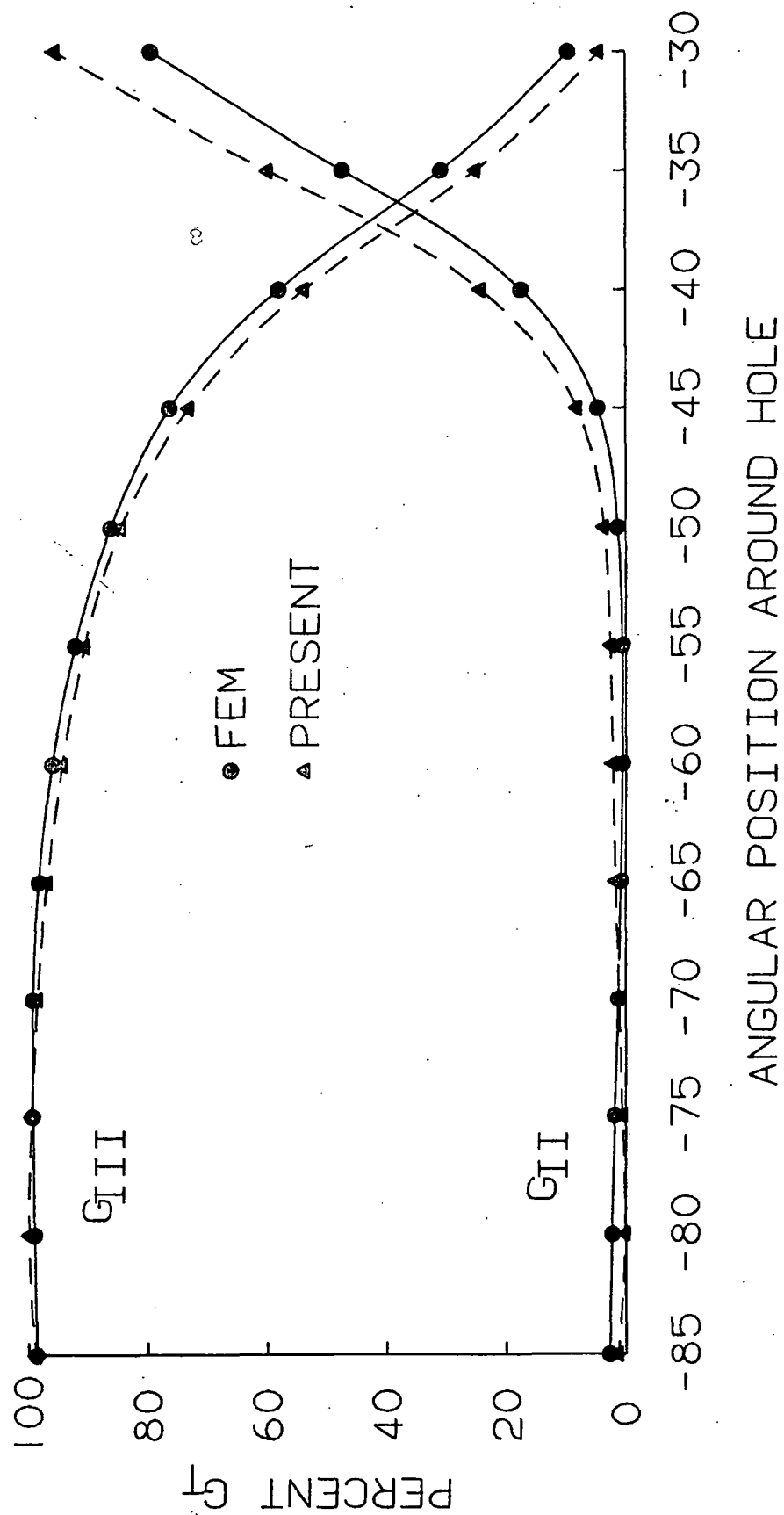


FIGURE 21. COMPARISON OF ENERGY RELEASE RATE FOR A  
[45/90/-45/0]<sub>s</sub> DELAMINATED AT -45/0 INTERFACE

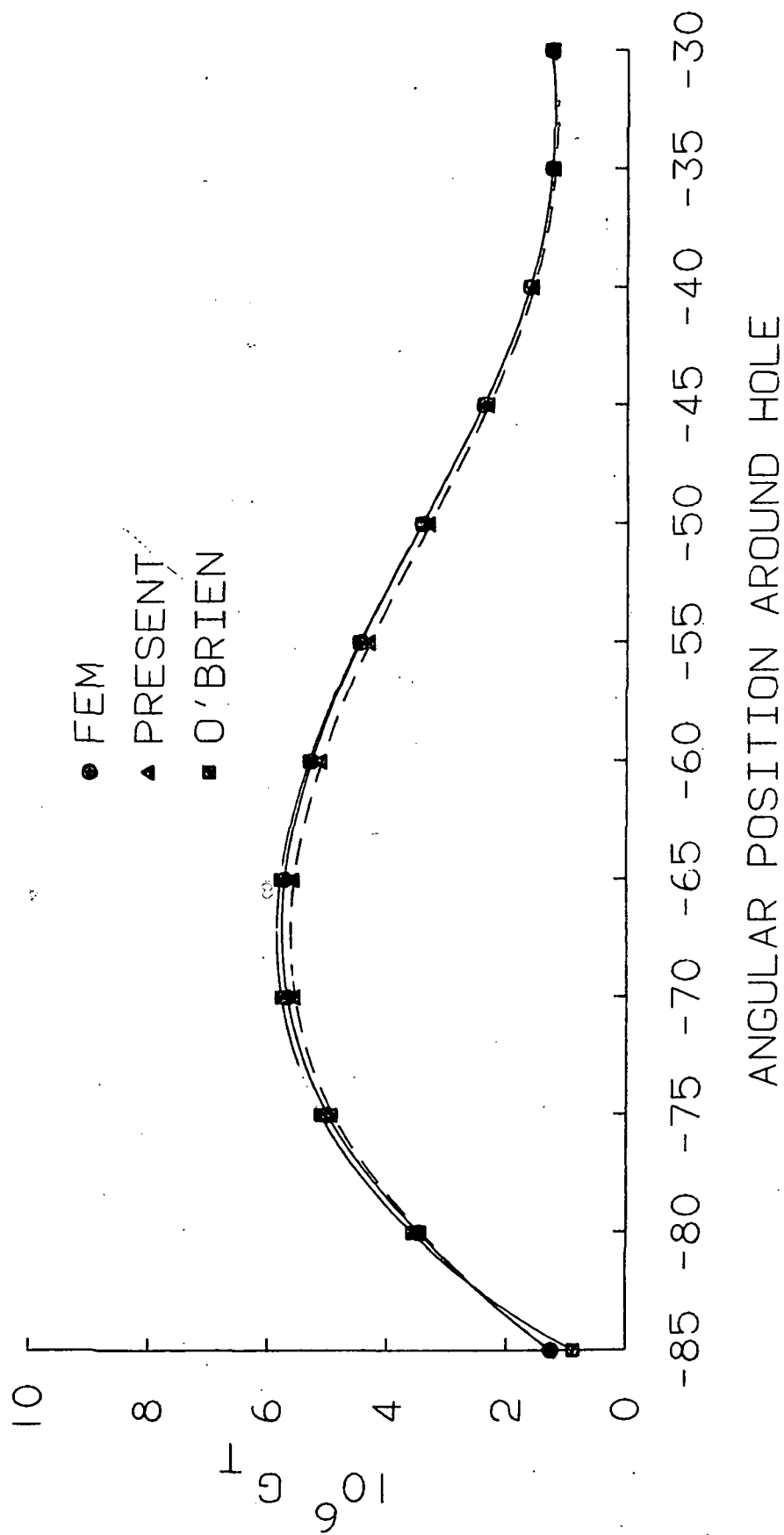


FIGURE 22. COMPARISON OF ENERGY RELEASE RATE FOR A  
[45/90/-45/0]<sub>s</sub> DELAMINATED AT -45/0 INTERFACE

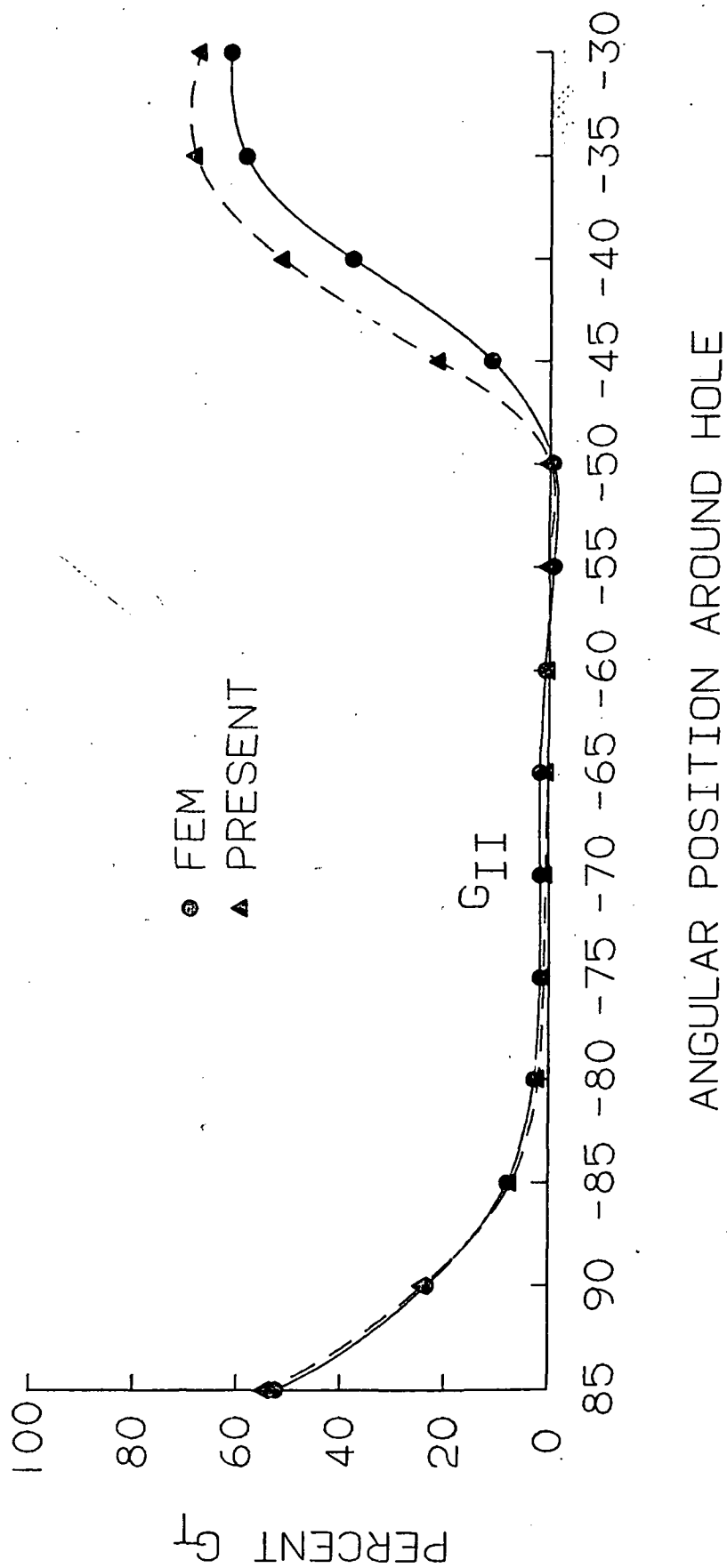


FIGURE 23.  $[45/90/-45/0]_s$  LAMINATE WITH CIRCULAR HOLE  
EDGE DELAMINATION AT  $90/-45$  INTERFACE

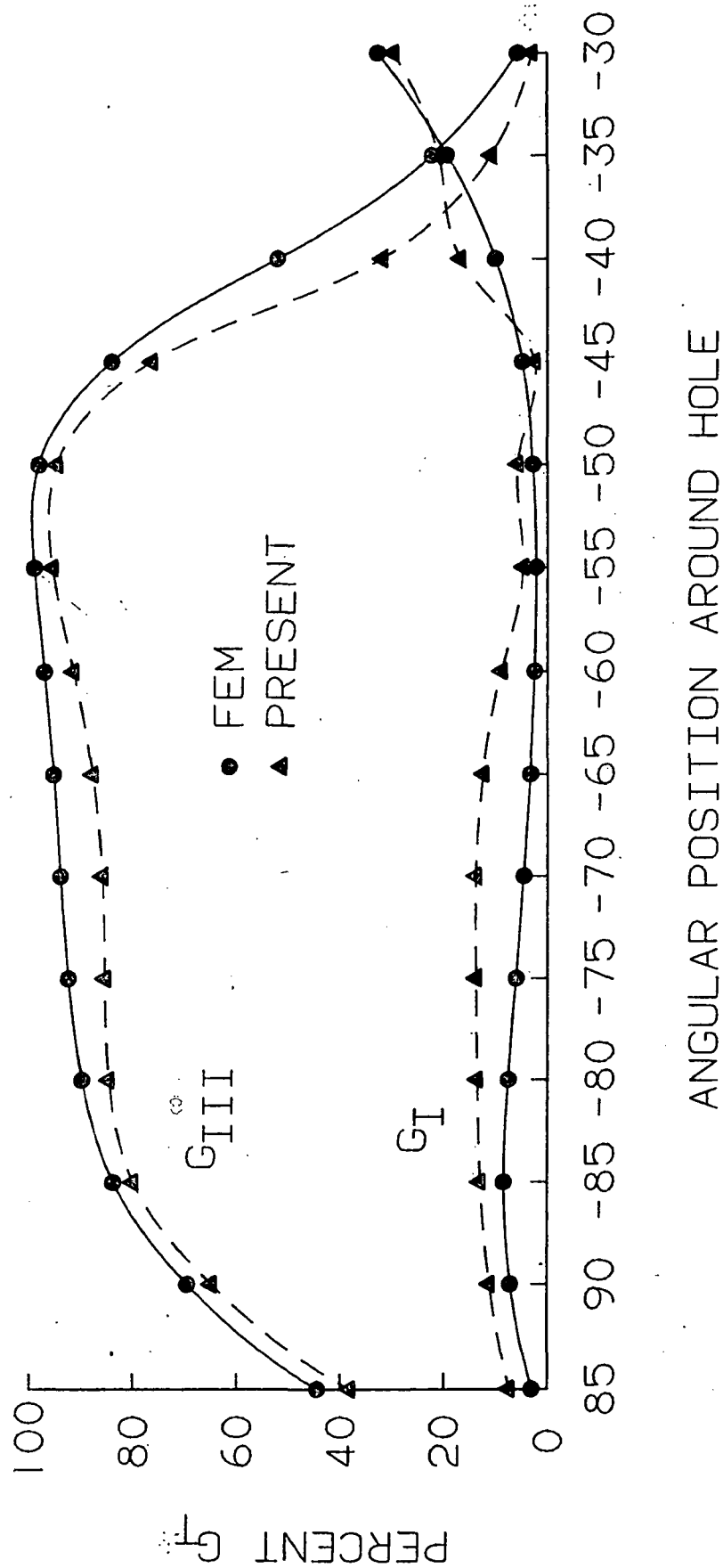


FIGURE 24.  $[45/90/-45/0]_s$  LAMINATE WITH CIRCULAR HOLE  
EDGE DELAMINATION AT  $90^\circ$ - $45^\circ$  INTERFACE

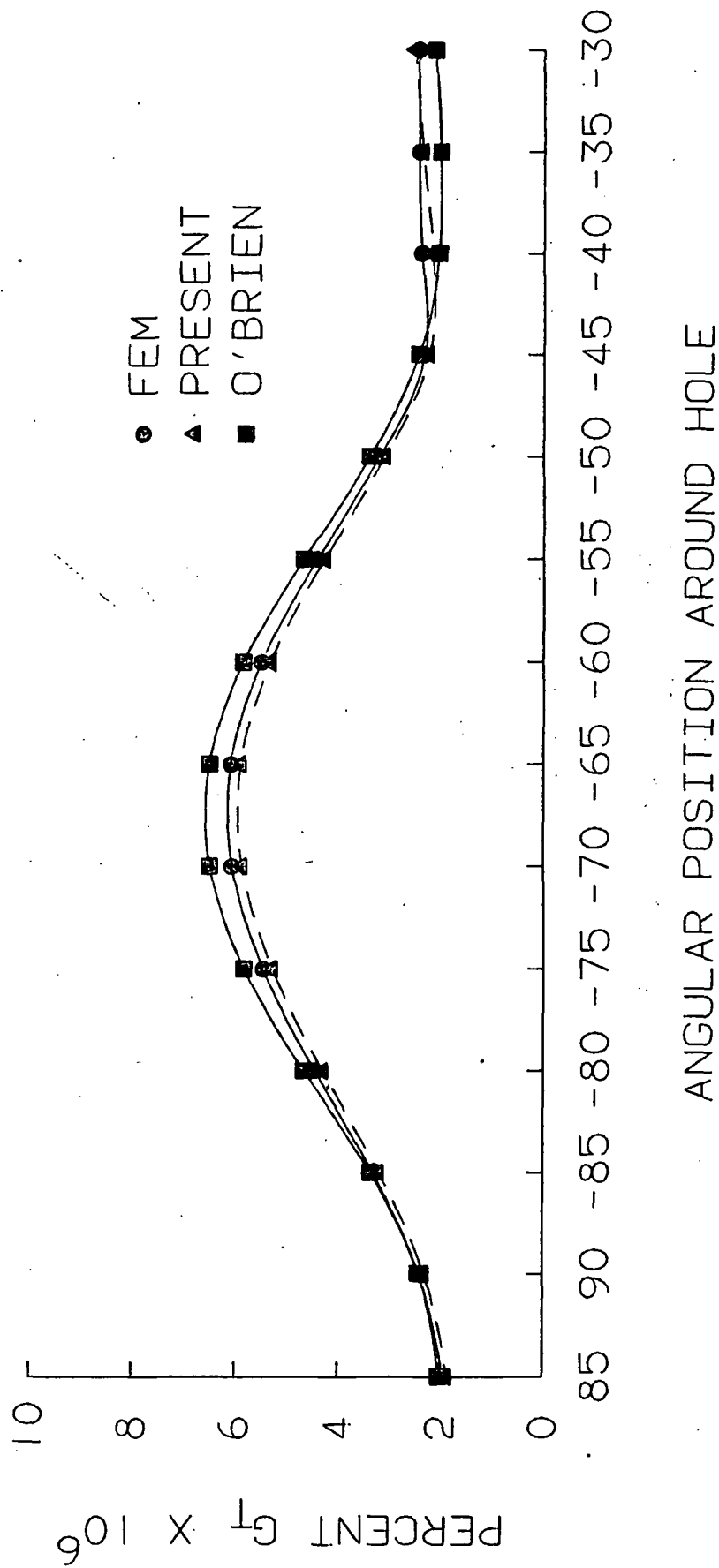
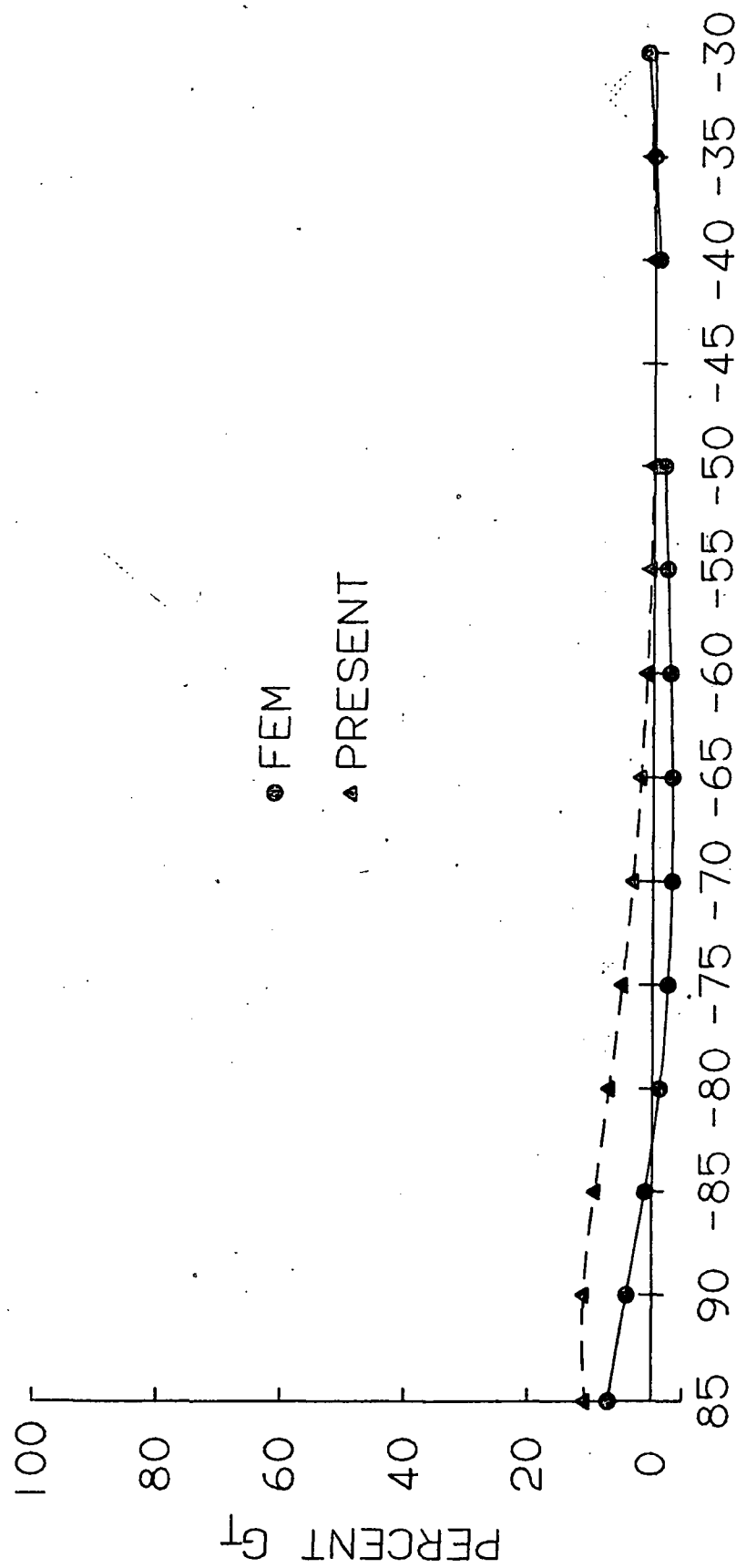
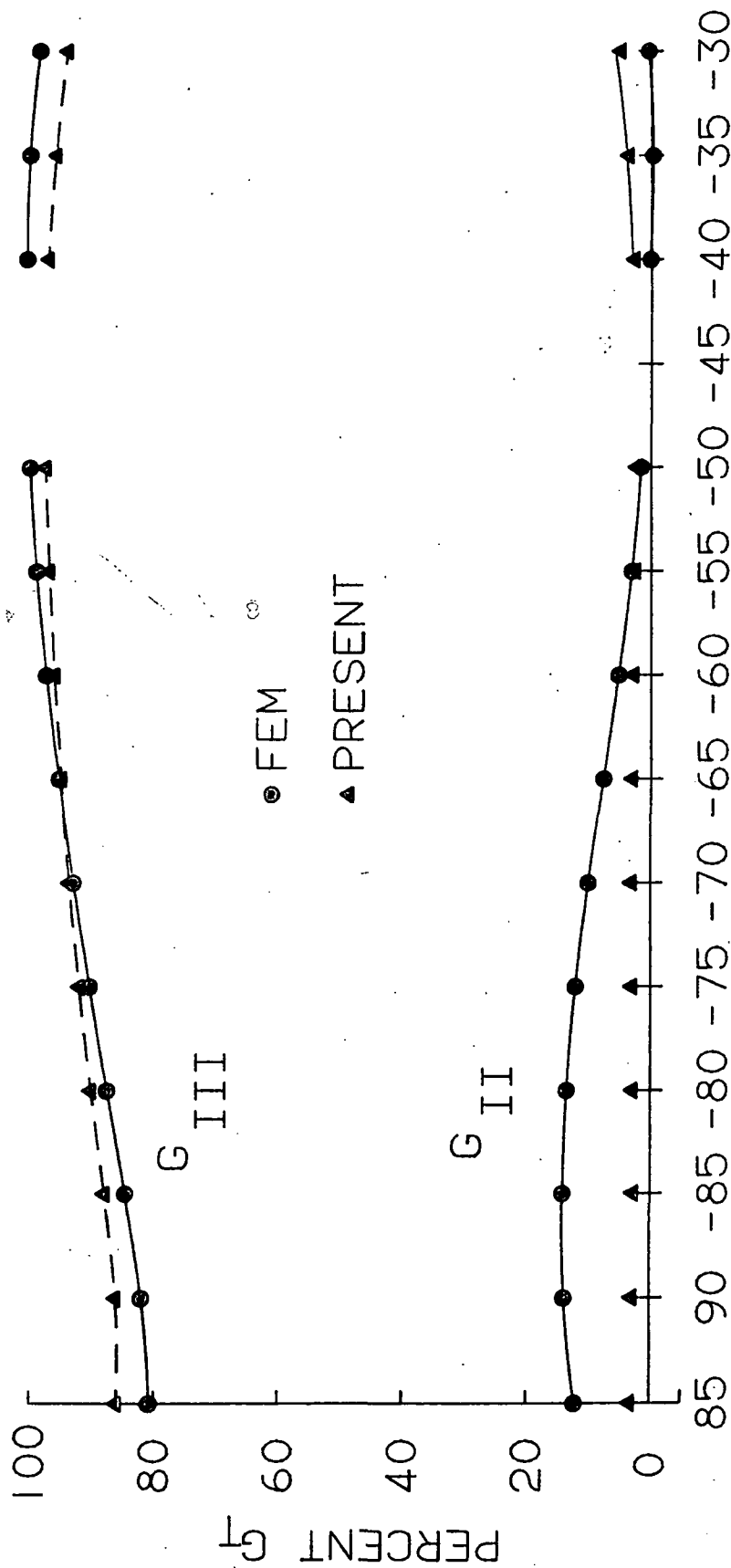


FIGURE 25. [45/90/-45/0]<sub>s</sub> LAMINATE WITH CIRCULAR HOLE  
EDGE DELAMINATION AT 90/-45 INTERFACE



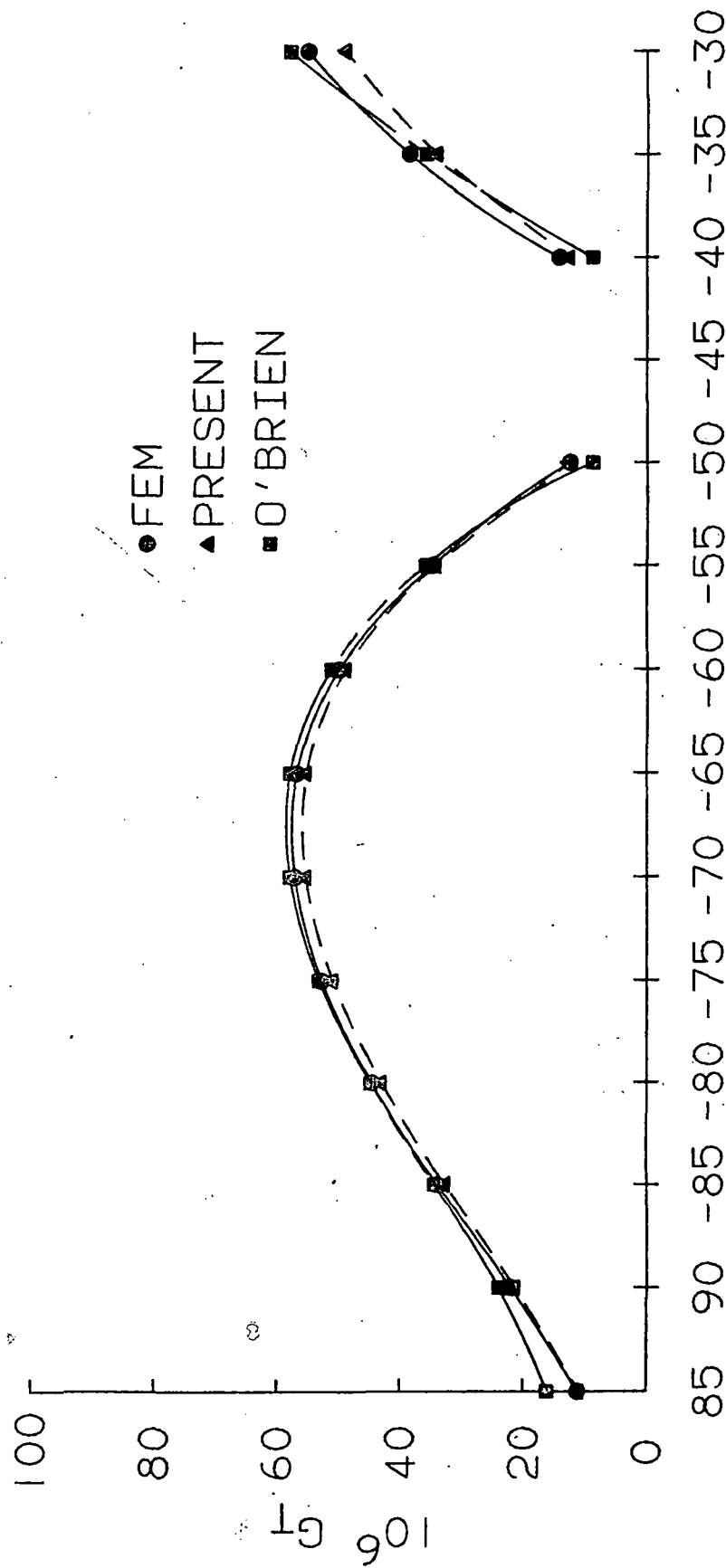
ANGULAR POSITION AROUND HOLE

FIGURE 26. COMPARISON OF ENERGY RELEASE RATE FOR A  
 [45/90/-45/0]<sub>s</sub> DELAMINATED AT -45/0 INTERFACE



ANGULAR POSITION AROUND HOLE

FIGURE 27. COMPARISON OF ENERGY RELEASE RATE FOR A  
[45/90/-45/0]<sub>s</sub> DELAMINATED AT 45/90 INTERFACE



ANGULAR POSITION AROUND HOLE

FIGURE 28. COMPARISON OF ENERGY RELEASE RATE FOR A  
[45/90/-45/0]<sub>s</sub> DELAMINATED AT 45/90 INTERFACE



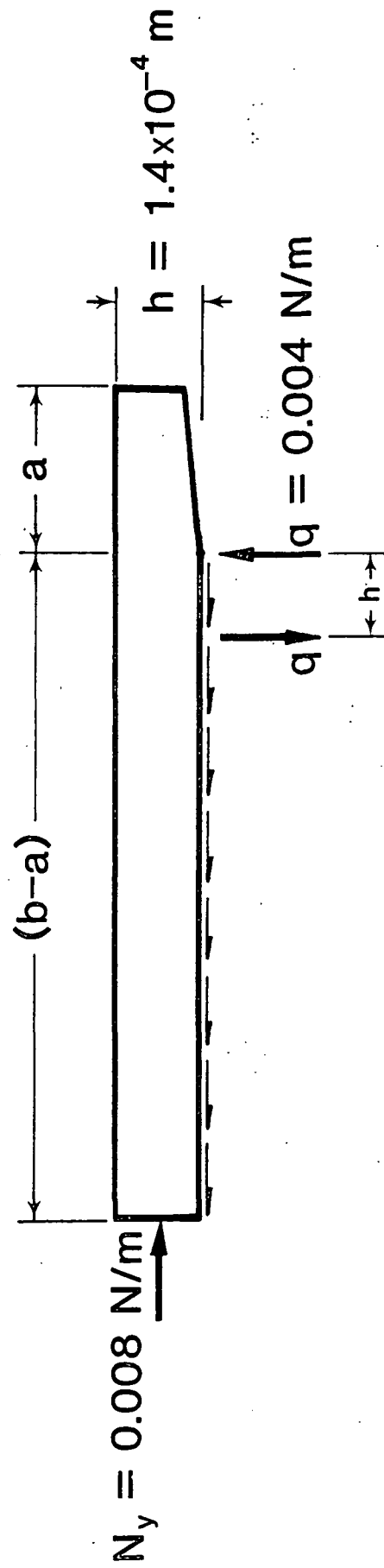


FIGURE 29. FREE BODY DIAGRAM OF THE TOP PLY IN A  $[0/45/90/-45]_s$  LAMINATE